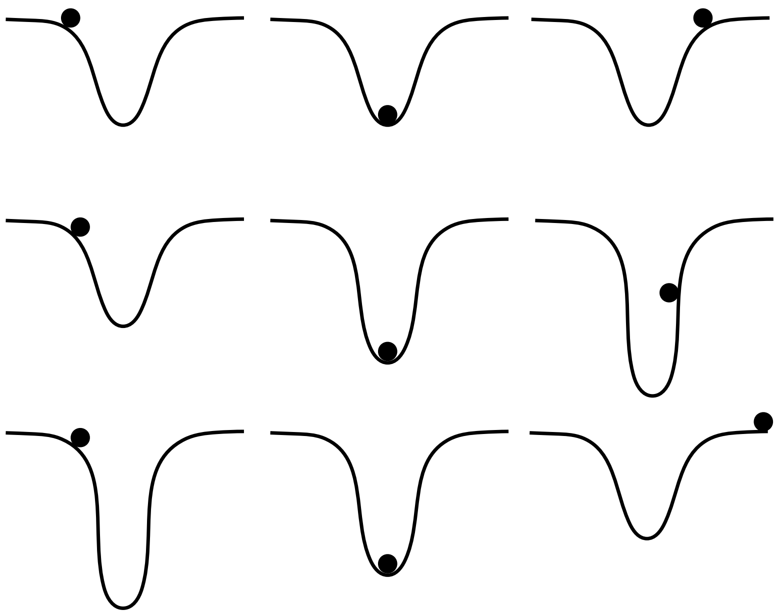
the other side and there is no relaxation. If the potential grows with time, the particle will need to expend more energy to cross it and will not have enough energy to get back out of the potential well, thus losing energy. If the potential shrinks with time, the particle will gain energy as it crosses the well.



As a galaxy or cluster forms, the gravitational potential changes significantly as mass accretes and collapses into a halo. Averaging over all particles, the timescale for violent relaxation t_{vr} is

$$\begin{aligned} t_{\mathrm{vr}} &= \left\langle \frac{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^2}{E^2} \right\rangle^{-1/2} \\ &= \left\langle \frac{\left(\frac{\partial \phi}{\partial t}\right)^2}{E^2} \right\rangle^{-1/2} \\ &\sim \left\langle \frac{\dot{\phi}^2}{\phi} \right\rangle^{-1/2} \end{aligned} \tag{77}$$

where in the last step we used the time-dependent virial theorem (see Lynden-Bell 1967). This occurs on roughly the same timescale as free-fall since this is the timescale at which the potential changes during collapse. It's very fast, hence 'violent' relaxation!

3 Modelling galaxies

So far, we have looked at the basic dynamical properties of galaxies. Now we discuss the main ingredients of modelling galaxies:

- potential-density pairs (the common potential)

- orbits (trajectories of stars orbiting in a potential)
- phase-space distribution function (distribution of orbits, Vlasov equation)
- stability (Jeans criterion)
- composition of stars (stellar populations), star formation rate, initial mass function
- chemical evolution of galaxies
- active galaxies

3.A Potential-density pairs

Stars move in a collective potential. What are interesting potentials and the related density functions?

Scalar potential:

$$-\vec{\nabla}\phi = \frac{1}{m}\vec{F} \tag{78}$$

Note that $m\phi = U$ is the potential energy of the system and using Poisson's equation $\nabla^2\phi = 4\pi G\rho$, we get

$$\phi(\vec{r}) = G\int \frac{\rho(\vec{r})}{|\vec{r'} - \vec{r}|}\mathrm{d}^3\vec{r} \tag{79}$$

$$\Rightarrow \text{potential } \phi - \text{density } \rho - \text{pairs!} \tag{80}$$

Examples:

- Kepler/point mass potential:

$$\phi = -\frac{GM}{r} \tag{81}$$

  To find $\vec{F}$, we take the gradient of ϕ

$$\frac{1}{m}\vec{F} = \frac{GM}{r^2}\hat{e}_r \ . \tag{82}$$

- Homogeneous sphere:

$$\begin{aligned} \rho(\vec{r}) &= \frac{M}{\frac{4}{3}\pi R^3} \\ \vec{F} &=? \end{aligned} \tag{83}$$

  We do not have ϕ, so we need a different way to get $\vec{F}$. We can use Gauss's theorem for gravity for a surface S_r with radius R enclosing a volume V_r:

$$\begin{aligned} \int_{S_r} \vec{F}\cdot\mathrm{d}\vec{S} &= \int_{V_r} (\vec{\nabla}\cdot\vec{F})\mathrm{d}V \\ &= -m\int_V (\vec{\nabla}^2\phi)\mathrm{d}V \\ &= -4\pi Gm\int \rho(\vec{r})\mathrm{d}V \\ &= -4\pi GM(<r)m \end{aligned} \tag{84}$$

Since we're working with gravity, we have $\vec{F}(r) = -F(r)\hat{e}_r$ and

$$\int_{S_r} \vec{F} \cdot \mathrm{d}\vec{S} = \int_{S_r} F(r)(-\hat{e}_r)\mathrm{d}\vec{S} = -4\pi r^2 F(r) \tag{85}$$

$$\Rightarrow 4\pi r^2 F(r) = r\pi G M(< r)m \tag{86}$$

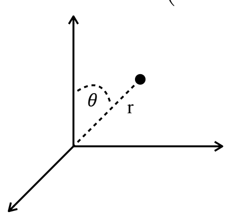
So

$$\begin{aligned} \text{Outside the sphere} : r > R \Rightarrow F(r) &= \frac{GMm}{r^2} \\ \text{Inside the sphere} : r < R \Rightarrow F(r) &= 4\pi G\frac{\rho r}{3}m \end{aligned} \tag{87}$$

From these, we can now also get $\phi : \frac{1}{m}\vec{F} = -\vec{\nabla}^2\phi$

$$\begin{aligned} \text{Outside the sphere} : r > R \Rightarrow \phi(r) &= \frac{GMm}{r} + \text{constant} \\ \text{Inside the sphere} : r < R \Rightarrow \phi(r) &= 2\pi G\frac{\rho r^2}{3}m + \text{constant} \end{aligned} \tag{88}$$

- Mestel disk (example of a disk potential):



$$\phi(r,\theta) = v_c^2\left[\ln\frac{r}{r_0} + \ln\frac{1+|\cos\theta|}{2}\right] \tag{89}$$

Is this a disk? It's hard to see based on the potential, so we need to find ρ. Let's look at Poisson's equation:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi}}_{=0,\ \text{since no } \varphi \text{ dependence}} \tag{90}$$

Using $\phi = v_c^2\phi_0$:

$$\begin{aligned} \nabla^2\phi &= \frac{v_c^2}{r^2}\frac{\partial}{\partial}\left(r^2\frac{1}{r}\right) + \frac{v_c^2}{r^2\sin\theta}\left(\cos\theta\frac{\partial\phi_0}{\partial\theta} + \sin\theta\frac{\partial^2\phi_0}{\partial\theta^2}\right) \\ &= \frac{v_c^2}{r^2}\left[1 + \left(\frac{\cos\theta}{\sin\theta}\frac{\partial\phi_0}{\partial\theta} + \frac{\partial^2\phi_0}{\partial\theta^2}\right)\right] \end{aligned} \tag{91}$$

We now calculate $\frac{\partial\phi_0}{\partial\theta}$ and $\frac{\partial^2\phi_0}{\partial\theta^2}$. We assume $\cos\theta > 0$. The calculations are the same or $\cos\theta < 0$ except for an overall sign change $\cos\theta \to -\cos\theta$.

$$\phi_0 = \ln\left(\frac{r}{r_0}\right) + \ln\left(\frac{1+\cos\theta}{2}\right) \tag{92}$$

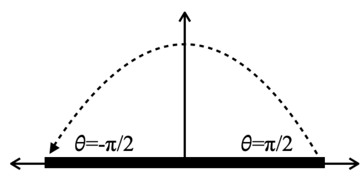
Then

$$\begin{aligned}\frac{\partial \phi}{\partial \theta} &= \frac{2}{1+\cos\theta}\left(-\frac{\sin\theta}{2}\right) = \left(-\frac{\sin\theta}{1+\cos\theta}\right)\\ \frac{\partial^2 \phi_0}{\partial \theta^2} &= -\frac{\cos\theta}{1+\cos\theta} - \frac{\sin^2\theta}{(1+\cos\theta)^2}\end{aligned} \tag{93}$$

So

$$\begin{aligned}\frac{\cos\theta}{\sin\theta}\frac{\partial \phi_0}{\partial \theta} + \frac{\partial^2 \phi_0}{\partial \theta^2} &= -\frac{2\cos\theta}{1+\cos\theta} - \frac{\sin^2\theta}{(1+\cos\theta)^2}\\ &= \frac{-2\cos\theta - 2\cos^2\theta - \sin^2\theta}{(1+\cos\theta)^2}\\ &= -\frac{1+\cos^2\theta + 2\cos\theta}{(1+\cos\theta)^2}\\ &= -\frac{(1+\cos\theta)^2}{(1+\cos\theta)^2}\\ &= -1\end{aligned} \tag{94}$$

For $\cos\theta \neq 0$, this gives $\nabla^2\phi = \frac{v_c^2}{r^2}(1-1) = 0$, so there is no density for $\theta \neq \pi/2$ and all mass is in a thin plane with infinite density ρ (3D density).



We can calculate the surface density

$$\Sigma(r) = \int_{-\infty}^{+\infty} \frac{1}{4\pi G}\vec{\nabla}^2\phi \,\mathrm{d}z \tag{95}$$

With $z = r\cos\theta$ so $\mathrm{d}z = -r\sin\theta \mathrm{d}\theta + \cos\theta dr \approx -r\, d\theta$ since $\theta \approx \pi/2$, we get

$$\begin{aligned}\Sigma(r) &= \int \rho \,\mathrm{d}z\\ &= \int_{\frac{\pi}{2}+\epsilon}^{\frac{\pi}{2}-\epsilon} \frac{1}{4\pi G}\vec{\nabla}^2\phi(-r\,\mathrm{d}\theta)\ .\end{aligned} \tag{96}$$

We go from $\frac{\pi}{2}+\epsilon$ where $z<0$ to $\frac{\pi}{2}-\epsilon$ where $z>0$. We can then switch the bounds and change the overall sign

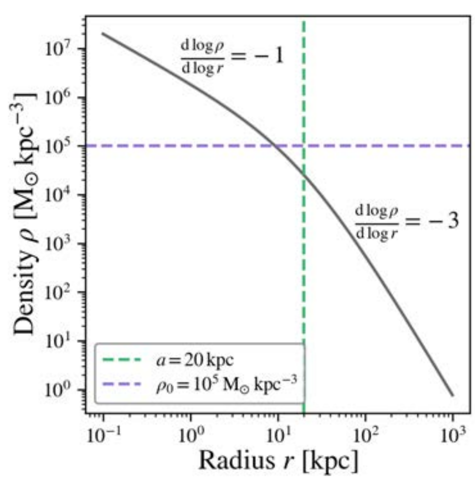
$$
\begin{aligned}
\Sigma(r) &= \frac{1}{4\pi G}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \vec{\nabla}^2\phi\, r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi G}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \frac{v_c^2}{r^2}\left(1+\left[\frac{\cos\theta}{\sin\theta}\frac{\partial\phi_0}{\partial\theta}+\frac{\partial^2\phi_0}{\partial\theta^2}\right]\right) r\,\mathrm{d}\theta \\
&\approx \frac{1}{4\pi G}\frac{v_c^2}{r}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial\phi_0}{\partial\theta}+\frac{\partial^2\phi_0}{\partial\theta^2}\right)\mathrm{d}\theta \\
&\text{(continuous functions} \to 0 \text{ for } \epsilon\to 0) \\
&\approx \frac{1}{4\pi G}\frac{v_c^2}{r}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon}\frac{\partial^2\phi_0}{\partial\theta^2}d\theta \\
&= \frac{1}{4\pi G}\left[\frac{\partial\phi_0}{\partial\theta}\right]_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon}
\end{aligned}
\tag{97}
$$

When $\theta > \frac{\pi}{2}$, $\cos\theta < 0$ and $|\cos\theta| = -\cos\theta$, and when $\theta < \frac{\pi}{2}$, $\cos\theta > 0$ and $|\cos\theta| = \cos\theta$. So we take the derivative using $-\cos\theta$ in the first term and $\cos\theta$ in the second term

$$
\begin{aligned}
\Sigma(r) &= \frac{1}{4\pi G}\left(\left[\frac{\sin\theta}{1-\cos\theta}\right]_{\frac{\pi}{2}+\epsilon}-\left[\frac{-\sin\theta}{1+\cos\theta}\right]_{\frac{\pi}{2}-\epsilon}\right) \\
&(\epsilon\to 0) \\
&= \frac{1}{4\pi G}\frac{v_c^2}{r}(1+1) \\
\Rightarrow \Sigma(r) &= \frac{1}{2\pi G}\frac{v_c^2}{r}\,.
\end{aligned}
\tag{98}
$$

- Navarro-Frenk-White profile (NFW):
  empirical profile found in simulations of CDM halos.

$$
\rho(r) = \frac{\rho_0}{\left(\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^2\right.} \propto \begin{cases} r^{-1} & r \ll a \\ r^{-3} & r \gg a \end{cases} \tag{99}
$$



Simulations showed the ρ_0 and a are strongly correlated for CDM halos, so halos are approximately members of a 1-parameter family. The conventional choice for this parameter is r_{200}, the distance which has an enclosed density 200 times the cosmic critical density ρ_c (which we will cover later) or $M_{200} = 200\rho_c \frac{4}{3}\pi r_{200}^3$.

The *concentration* of a halo is

$$\boxed{c = \frac{r_{200}}{a}} \tag{100}$$

Central result:
The second parameter c is only a very weak function of mass and for fixed mass, and it is the same for all halos in that mass range.

$$\phi = -4\pi\rho_0 a^2 \frac{\ln\left(1+\frac{r}{a}\right)}{\frac{r}{a}} + \text{constant} \tag{101}$$

Related topics:

- Core-cusp problem: From observations of stellar dynamics, the inner profile of halos flattens to a slope ~ 0 (core) instead of -1 (cusp). This is possibly due to supernova feedback, but it could also be resolved through modifications of cold dark matter.
- Diversity of shapes problem: Observationally, halos display diversity in the shapes of their profiles with some cuspier and some more cored profiles whereas, in simulations, halos are universally described by the NFW profile and self-similar across mass ranges (the profiles look the same when scaled).
- Missing satellite problem: Simulations produce more satellite halos than there are observed satellite galaxies. It's possible that not all subhalos form stars, so we need to be able to find "dark subhalos." This could be done by looking for disruptions in stellar streams or through gravitational lensing. Recently, however, there have been many more satellites found as our observational techniques improve.
- Too-big-to-fail problem: This is related to the missing satellites problem, where the number of predicted large halos doesn't match the number of large galaxies observed (but the total number of satellite halos is consistent). The gravitational potential of these galaxies, however, is large enough that they should have collected enough gas and stars to form galaxies and maintain their evolution (e.g. not lose the stars through stripping).

3.B Orbits

Now that we have looked at potential-density pairs, we can study orbits in these potentials. *Orbits* refer to the motion of stars through 6D phase space $(\vec{x}(t), \vec{v}(t))$. Often, the integrals of motion restrict the dimensionality of the orbit (1 per integral of motion).

Integrals of motion:
The orbital energy E is:

$$E = \frac{1}{2}v^2 + \phi(r) = \frac{1}{2}\dot{r}^2 + \phi(r) \tag{102}$$