parameter is r_{200}, the distance which has an enclosed density 200 times the cosmic critical density ρ_c (which we will cover later) or $M_{200} = 200\rho_c \frac{4}{3}\pi r_{200}^3$.

The *concentration* of a halo is

$$\boxed{c = \frac{r_{200}}{a}} \tag{100}$$

Central result:
The second parameter c is only a very weak function of mass and for fixed mass, and it is the same for all halos in that mass range.

$$\phi = -4\pi\rho_0 a^2 \frac{\ln\left(1+\frac{r}{a}\right)}{\frac{r}{a}} + \text{constant} \tag{101}$$

Related topics:

- Core-cusp problem: From observations of stellar dynamics, the inner profile of halos flattens to a slope ~ 0 (core) instead of -1 (cusp). This is possibly due to supernova feedback, but it could also be resolved through modifications of cold dark matter.
- Diversity of shapes problem: Observationally, halos display diversity in the shapes of their profiles with some cuspier and some more cored profiles whereas, in simulations, halos are universally described by the NFW profile and self-similar across mass ranges (the profiles look the same when scaled).
- Missing satellite problem: Simulations produce more satellite halos than there are observed satellite galaxies. It's possible that not all subhalos form stars, so we need to be able to find "dark subhalos." This could be done by looking for disruptions in stellar streams or through gravitational lensing. Recently, however, there have been many more satellites found as our observational techniques improve.
- Too-big-to-fail problem: This is related to the missing satellites problem, where the number of predicted large halos doesn't match the number of large galaxies observed (but the total number of satellite halos is consistent). The gravitational potential of these galaxies, however, is large enough that they should have collected enough gas and stars to form galaxies and maintain their evolution (e.g. not lose the stars through stripping).

3.B Orbits

Now that we have looked at potential-density pairs, we can study orbits in these potentials. *Orbits* refer to the motion of stars through 6D phase space $(\vec{x}(t), \vec{v}(t))$. Often, the integrals of motion restrict the dimensionality of the orbit (1 per integral of motion).

Integrals of motion:
The orbital energy E is:

$$E = \frac{1}{2}v^2 + \phi(r) = \frac{1}{2}\dot{r}^2 + \phi(r) \tag{102}$$

Taking the time derivative (for a time-independent potential) gives us

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{r}\ddot{r} + \frac{\mathrm{d}\phi}{\mathrm{d}r}\dot{r} = \dot{r}\ddot{r} - \ddot{r}\dot{r} = 0 \tag{103}$$

which implies that the energy is constant along the orbit.

The angular momentum $\vec{L}$ (for a central force potential) is:

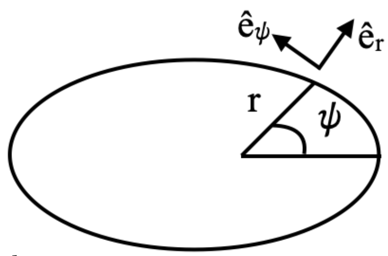
$$\vec{L} = \vec{r} \times \dot{\vec{r}} \tag{104}$$

and the time derivative is

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times (F(r)\hat{e}_r) = 0 \tag{105}$$

so angular momentum is also constant along the orbit. This means we have a 4D phase space instead of 6D for time-independent, central force potentials, which is often a good approximation.

Central potentials: $\phi = \phi(r)$
Goal: derive equations for radial and tangential components, which is sufficient to describe motion since it is a 4D phase space.



$$\begin{aligned} \hat{e}_r &= \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \\ \hat{e}_\psi &= \begin{pmatrix} \sin\psi \\ -\cos\psi \end{pmatrix} \end{aligned} \tag{106}$$

We have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} &= \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{e}_r) \\ &= \dot{r}\hat{e}_r + r\frac{\mathrm{d}}{\mathrm{d}t}(\hat{e}_r) \\ &= \dot{r}\hat{e}_r + r\left(\underbrace{\frac{\mathrm{d}\hat{e}_r}{\mathrm{d}r}}_{=0}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\mathrm{d}\hat{e}_r}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) \\ &= \dot{r}\hat{e}_r + r\dot{\psi}\underbrace{\frac{\mathrm{d}}{\mathrm{d}\psi}\hat{e}_r}_{=\frac{\mathrm{d}}{\mathrm{d}\psi}\begin{pmatrix} \cos\psi \\ sin\psi \end{pmatrix} = \begin{pmatrix} -\sin\psi \\ cos\psi \end{pmatrix} = \hat{e}_\psi} \\ &= \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi \end{aligned} \tag{107}$$

so

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} &= \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi) \\
&= (\ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi) + \frac{\mathrm{d}}{\mathrm{d}t}(r\dot{\psi}\hat{e}_\psi) \\
&= \ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi + \dot{r}(\dot{\psi}\hat{e}_\psi) + r\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\psi}\hat{e}_\psi\right) \\
&= \ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi + \dot{r}\dot{\psi}\hat{e}_\psi + r\ddot{\psi}\hat{e}_\psi + r\dot{\psi}\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\psi}_{\substack{= \frac{\mathrm{d}\hat{e}_\psi}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}t} \\ = -\dot{\psi}\hat{e}_r}} \\
&= \ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi + \dot{r}\dot{\psi}\hat{e}_\psi + r\ddot{\psi}\hat{e}_\psi - r\dot{\psi}^2\hat{e}_r \\
&= \left(\ddot{r} - r\dot{\psi}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\psi} + r\ddot{\psi}\right)\hat{e}_\psi
\end{aligned} \tag{108}$$

and, since we are using a central force,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} = F(r)\hat{e}_r \tag{109}$$

where F is the force per unit mass. Combining the two above equations, we get the scalar equations for 4D orbits for the radial and tangential components of motion:

$$\begin{aligned}
\text{radial} : \ddot{r} - r\dot{\psi}^2 &= F(r) \\
\text{tangential} : 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0\ .
\end{aligned} \tag{110}$$

For now, we focus on the radial equation and substitute $u = \frac{1}{r}$ to avoid the singularity at $r = 0$. Then

$$F(r) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{u}\right) - \frac{1}{u}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2\ . \tag{111}$$

With

$$\vec{L} = \vec{r} \times \vec{v} \Rightarrow L = r^2\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{112}$$

we can parameterize t with ψ to get $u = u(\psi)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{L}{r^2}\frac{\mathrm{d}}{\mathrm{d}\psi} = Lu^2\frac{\mathrm{d}}{\mathrm{d}\psi} \tag{113}$$

Then

$$\begin{aligned}
\vec{L} &= \vec{r} \times \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} \\
&= \vec{r} \times \left(\dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi\right) \\
&= r\hat{e}_r \times \left(\dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi\right) \\
&= r^2\dot{\psi} \\
&= r^2\frac{\mathrm{d}\psi}{\mathrm{d}t}\ .
\end{aligned} \tag{114}$$

Then we get

$$\begin{aligned} F(u) &= Lu^2 \frac{\mathrm{d}}{\mathrm{d}\psi}\left(Lu^2 \frac{\mathrm{d}}{\mathrm{d}\psi}\frac{1}{u}\right) - \frac{1}{u}\left(Lu^2 \frac{\mathrm{d}\psi}{\mathrm{d}\psi}\right) \\ &= Lu^2 \frac{\mathrm{d}}{\mathrm{d}\psi}\left(Lu^2 \frac{-1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\psi}\right) - \frac{1}{u}\left(Lu^2\right)^2 \\ &= -L^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\psi^2} - L^2u^3 \end{aligned} \tag{115}$$

which gives us the orbit equation:

$$\boxed{\frac{\mathrm{d}^2u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2u^2}} \tag{116}$$

with $u = u(\psi)$. Note that there is no time dependence.

We now examine some examples using this equation.

Examples:

- Kepler:

$$\phi(r) = -\frac{GM}{r} \rightarrow F(r) = -\frac{\mathrm{d}}{\mathrm{d}r}\phi = -\frac{GM}{r^2} = -GMu^2 \tag{117}$$

  so we get the orbital equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2u^2} = \frac{GM}{L^2} \; . \tag{118}$$

  Note that this is a harmonic oscillator, so we know the solution:

$$u(\psi) = C\, cos(\psi - \psi_0) + \frac{GM}{L^2} \tag{119}$$

  Then for $\psi = 0$ to $\psi = 2\pi$, $u(\psi = 0) = u(\psi = 2\pi)$ and we get closed orbits (frequency $\omega = 1$).

- Post-Newtonian relativistic correction:

$$\phi(r) = -\frac{GM}{r}\left(1 + \frac{2GM}{rc^2}\right) \tag{120}$$

  so

$$\begin{aligned} F(r) &= -\frac{\mathrm{d}}{\mathrm{d}r}\phi \\ &= -\frac{GM}{r^2} - \frac{4G^2M^2}{r^3c^2} \\ &= -GMu^2 - \frac{4G^2M^2}{c^2}u^3 \; . \end{aligned} \tag{121}$$

So we get the orbital equation:

$$\begin{aligned}
&\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2 u^2} = \frac{GM}{L^2} + \frac{4G^2M^2}{L^2c^2}u \\
\Rightarrow &\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + \left(1 - \underbrace{\frac{4G^2M^2}{L^2c^2}}_{\text{constant}}\right) u = \frac{GM}{L^2}\ .
\end{aligned} \tag{122}$$

The term in parentheses implies that $k^2 \neq 1$, so $\omega < 1$, which means that $u(\psi = 0) \neq u(\psi = 2\pi)$. This accounts for the precession of Mercury.

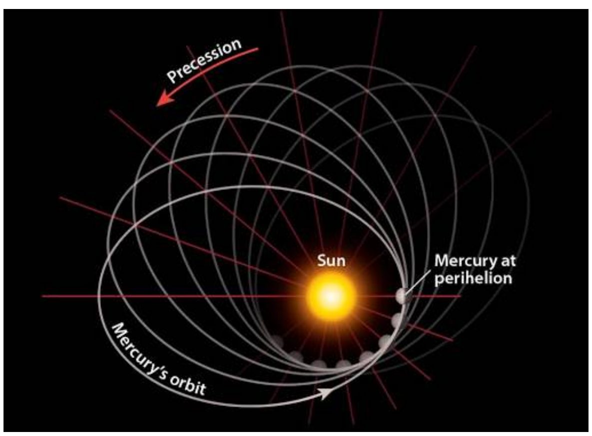
The solution for a harmonic oscillator

$$m\ddot{u} + k\dot{u} = \text{const} \tag{123}$$

is

$$u = \cos(\omega t - \phi) \tag{124}$$

where $\omega^2 = k/m$. Then if $w \neq 1$ and $k \neq 1$, $u(0) \neq u(2\pi)$. Here, t is analogous to ψ, so if the position u after one orbit when $\psi = 2\pi$ is not the same as when $\psi = 0$, the mass has not returned to its previous position and the orbit is not closed.



Axisymmetric Potentials: $\phi = \phi(R, |z|)$
We will derive equations for R, z, and ψ.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(r\hat{e}_r + z\hat{e}_z\right) \\
&= \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi + \dot{z}\hat{e}_z + z\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\hat{e}_z)}_{=0 \text{ since } \hat{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix}} \\
&= \dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi + \dot{z}\hat{e}_z\ .
\end{aligned} \tag{125}$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_r &= \dot{\psi}\hat{e}_\psi \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_\psi &= -\dot{\psi}\hat{e}_r \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_z &= 0\end{aligned} \tag{126}$$

then

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{e}_r + r\dot{\psi}\hat{e}_\psi + \dot{z}\hat{e}_z\right) \\ &= \ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi + \dot{r}\dot{\psi}\hat{e}_\psi + r\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\psi}\hat{e}_\psi\right) + \ddot{z}\hat{e}_z \\ &= \ddot{r}\hat{e}_r + \dot{r}\dot{\psi}\hat{e}_\psi + \dot{r}\dot{\psi}\hat{e}_\psi + r\ddot{\psi}\hat{e}_\psi - r\dot{\psi}^2\hat{e}_r + \ddot{z}\hat{e}_z \\ &= \left(\ddot{r} - r\dot{\psi}^2\right)\hat{e}_r + \underbrace{\left(2\dot{r}\dot{\psi} + r\ddot{\psi}\right)}_{=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\psi}\right)}\hat{e}_\psi + \ddot{z}\hat{e}_z\end{aligned} \tag{127}$$

and for an axisymmetric potential

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} = \vec{F} = \left(-\frac{\partial\phi}{\partial r}, 0, -\frac{\partial\phi}{\partial z}\right) \tag{128}$$

so we get each component of the force:

$$\begin{aligned}\text{radial} : \ddot{r} - r\dot{\psi}^2 &= -\frac{\partial\phi}{\partial r} \\ \text{tangential} : \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\psi}\right) &= 0 \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\psi}\right) &= 0 \\ &\text{(using conservation of } L_z = r^2\psi = \text{constant)} \\ \text{vertical} : \ddot{z} &= -\frac{\partial\phi}{\partial z}\ .\end{aligned} \tag{129}$$

We then rewrite this in terms of the effective potential:

$$\phi_{\text{eff}} = \phi + \frac{L_z^2}{2r^2} \tag{130}$$

where the last term is the centrifugal barrier. Since

$$\vec{v} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r} \tag{131}$$

then, using the above from $\dot{\vec{r}}$,

$$\begin{aligned}E &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\psi}^2 + \dot{z}^2\right) + \phi \\ &= \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \phi_{\text{eff}}\end{aligned} \tag{132}$$

so

$$
\begin{aligned}
\ddot{r} = r\dot{\psi}^2 = \frac{\partial \phi}{\partial r} \\
&= r\dot{\psi}^2 - \frac{\partial}{\partial r}\left(\phi_{\text{eff}} - \frac{L_z^2}{2r^2}\right) \\
&= r\dot{\psi}^2 - \frac{\partial \psi_{\text{eff}}}{\partial r} - \frac{L_z^2}{r^3} \\
&= r\dot{\psi}^2 - \frac{\partial \phi_{\text{eff}}}{\partial r} - \frac{r^4\dot{\psi}^2}{r^3} \text{ (using } L_z = r^2\dot{\psi}) \\
&= -\frac{\partial \phi_{\text{eff}}}{\partial r}
\end{aligned} \tag{133}
$$

and

$$
\ddot{z} = -\frac{\partial \phi_{\text{eff}}}{\partial z} \ . \tag{134}
$$

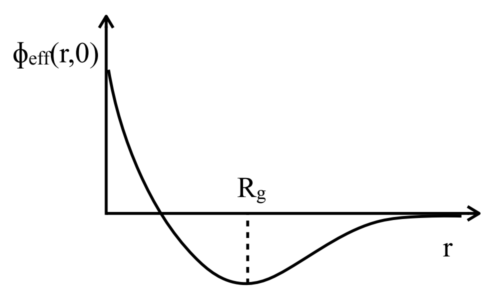
So finally we get the scalar equations for 4D orbits (E, L_z):

$$
\begin{aligned}
\ddot{r} &= -\frac{\partial \phi_{\text{eff}}}{\partial r} \\
\ddot{z} &= -\frac{\partial \phi_{\text{eff}}}{\partial z} \\
E &= \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \phi_{\text{eff}} \ .
\end{aligned} \tag{135}
$$

Note that the orbits have a uniform rotation around the symmetry axis (z) with $\dot{\psi} = \frac{L_z}{r^2}$, but we have oscillations in r and z. If r is not oscillating, then $z = 0$, and any perturbation leads to oscillations in z and r.

Guiding center and circular orbits:
ϕ_{eff} has a minimum at some R_g such that for a given L_z, $\phi(R_g, 0)$ is minimal:
At the minimum:

$$
\begin{aligned}
\left.\frac{\partial \phi_{\text{eff}}}{\partial r}\right|_{r=R_g, z=0} &= 0 \\
\left.\frac{\partial \phi_{\text{eff}}}{\partial z}\right|_{r=R_g, z=0} &= 0
\end{aligned} \tag{136}
$$



where symmetry implies that there is no force at $z = 0$.
Then we get:

$$
\begin{aligned}
0 &= \left.\frac{\partial \phi_{\text{eff}}}{\partial r}\right|_{R_g,0} \\
\Rightarrow \left.\frac{\partial \phi_{\text{eff}}}{\partial r}\right|_{R_g,0} &= \frac{L_z^2}{R_g^3} = R_g\dot{\psi}^2
\end{aligned} \tag{137}
$$

since

$$
\frac{\partial \phi_{\text{eff}}}{\partial r} = \frac{\partial \phi}{\partial r} - \frac{L_z^2}{r^3} \ . \tag{138}
$$

We also have

$$\Omega^2 = \dot{\psi}^2 = \frac{L_z^2}{R_g^4} \,. \tag{139}$$

And since

$$\left.\frac{\partial \phi_{\text{eff}}}{\partial r}\right|_{R_g,0} = \left.\frac{\partial \phi_{\text{eff}}}{\partial z}\right|_{R_g,0} = 0 \tag{140}$$

then

$$\ddot{r} = \ddot{z} = 0 \tag{141}$$

so we have a circular orbit with speed $\Omega = \dot{\psi}$. The minimum of ϕ_{eff} occurs at a radius R_g at which a circular orbit has angular momentum L_z and $E = \phi_{\text{eff}}$. This orbit is called the *guiding center*. If an object is pushed off the guiding center, there is a restoring force that leads to oscillations, or *epicycles*.

Epicycle approximation:
In disk galaxies, many stars are on mostly circular orbits, but they are not exactly circular. We look for small perturbations around the circular orbit.
We will define our coordinate system (x, y) as

$$\begin{aligned} x &= r - R_g \\ y &= z \end{aligned} \tag{142}$$

and expand ϕ_{eff} around $(x, y) = (0, 0)$.
Keeping only second-order terms for the epicycle approximation, we get

$$\tilde{\phi}_{\text{eff}}(x,y) = \tilde{\phi}_{\text{eff}}(0,0) + (\tilde{\phi}_{\text{eff},x})x + (\tilde{\phi}_{\text{eff},y})y + (\tilde{\phi}_{\text{eff},xy})xy + \frac{1}{2}(\tilde{\phi}_{\text{eff},xx})x^2 + \frac{1}{2}(\tilde{\phi}_{\text{eff},yy})y^2 + \ldots \tag{143}$$

where

$$\begin{aligned} \tilde{\phi}_{\text{eff},x} &= \left.\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial x}\right|_{0,0} = 0 \\ \tilde{\phi}_{\text{eff},y} &= \left.\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial y}\right|_{0,0} = 0 \\ \tilde{\phi}_{\text{eff},xy} &= \left.\frac{\partial^2 \tilde{\phi}_{\text{eff}}}{\partial x \partial y}\right|_{0,0} = 0 \qquad \text{(by symmetry).} \end{aligned} \tag{144}$$

We define κ and ν:

$$\begin{aligned} \kappa^2 &\equiv \tilde{\phi}_{\text{eff},xx} = \phi_{\text{eff},rr} = \left.\frac{\partial^2 \phi_{\text{eff}}}{\partial r^2}\right|_{R_g,0} \\ \nu^2 &\equiv \tilde{\phi}_{\text{eff},yy} = \phi_{\text{eff},zz} = \left.\frac{\partial^2 \phi_{\text{eff}}}{\partial z^2}\right|_{R_g,0} \,. \end{aligned} \tag{145}$$

so

$$\begin{aligned} \tilde{\phi}_{\text{eff}} &\approx \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 y^2 + \tilde{\phi}_{\text{eff}}(0,0) \\ &= \frac{1}{2}\left(\kappa x\right)^2 + \frac{1}{2}\left(\nu y\right)^2 + \tilde{\phi}_{\text{eff}}(0,0) \,. \end{aligned} \tag{146}$$

We can write down the equations of motion for

$$\tilde{\phi}_{\text{eff}}(x, y) = \tilde{\phi}_{\text{eff}}(0, 0) + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 y^2 \tag{147}$$

so

$$\begin{aligned} \ddot{x} = \ddot{r} = -\frac{\partial \phi_{\text{eff}}}{\partial r} &= -\frac{\partial x}{\partial r}\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial x} = -\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial x} = -\kappa^2 x \\ \ddot{y} = \ddot{z} = -\frac{\partial \phi_{\text{eff}}}{\partial z} &= -\frac{\partial y}{\partial z}\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial y} = -\frac{\partial \tilde{\phi}_{\text{eff}}}{\partial y} = -\nu^2 y \end{aligned} \tag{148}$$

and we get the final equations of motion:

$$\boxed{\begin{aligned} \ddot{x} &= -\kappa^2 x \\ \ddot{y} &= -\nu^2 y \end{aligned}}\,. \tag{149}$$

This is harmonic oscillation with epicycle frequency κ and vertical frequency ν in addition to the circular frequency

$$\Omega(r) = \frac{L_z}{r^2} = \frac{v_c}{r} = \sqrt{\frac{1}{r}\frac{\partial \phi}{\partial r}} \tag{150}$$

where for the last equality we used the fact that the centripetal force is equal to the gravitational force $\frac{v_c^2}{r} = \frac{\partial \phi}{\partial r}$ on a circular orbit.

At $r = R_g$ and $\Omega(R_g)$, we can rewrite κ in terms of Ω:

$$\begin{aligned}
&\left(r\frac{\mathrm{d}\Omega^2}{\mathrm{d}r} + 4\Omega^2 \right) \Bigg|_{R_g} \\
&= \left(r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right) + 4\frac{L_z^2}{r^4} \right) \Bigg|_{R_g} \\
&= \left(r\left(-\frac{1}{r^2}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial r^2}\right) + 4\frac{L_z^2}{r^4} \right) \Bigg|_{R_g} \\
&= \left(-\underbrace{\frac{1}{r}\frac{\partial\phi}{\partial r}}_{=\Omega^2 = \frac{L_z^2}{r^4}} + \frac{\partial^2\phi}{\partial r^2} + 4\frac{L_z^2}{r^4} \right) \Bigg|_{R_g} \\
&= \left(-\frac{L_z^2}{r^4} + \frac{\partial^2\phi}{\partial r^2} + 4\frac{L_z^2}{r^4} \right) \Bigg|_{R_g} \\
&= \left(\frac{\partial^2\phi}{\partial r^2} + 3\frac{L_z^2}{r^4} \right) \Bigg|_{R_g} \\
&= \left(\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r} - \frac{L_z^2}{r^3}\right) \right) \Bigg|_{R_g} \\
&= \left(\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\underbrace{\left(\phi + \frac{L_z^2}{2r^2}\right)}_{=\phi_{\text{eff}}}\right) \right) \Bigg|_{R_g} \\
&= \frac{\partial^2\phi_{\text{eff}}}{\partial r^2}\Bigg|_{R_g} \\
&= \kappa^2 \\
\Rightarrow \kappa^2 &= \left(r\frac{\mathrm{d}(\Omega^2)}{\mathrm{d}r} + 4\Omega^2 \right) \ .
\end{aligned} \tag{151}$$

Motion in the epicycle approximation (valid for $x, y, z \ll R_g$):
We look at each component of the motion:

$$\begin{aligned}
\text{radial} : r(t) &= r_0 \cos(\kappa t + \alpha) + R_g \\
\text{vertical} : z(t) &= z_0 \cos(\nu t + \beta) \\
\text{tangential} : \dot{\psi} &= \frac{L_z}{r^2} = \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} = \Omega(R_g)\left(1 + \frac{x}{R_g}\right)^{-2}
\end{aligned} \tag{152}$$

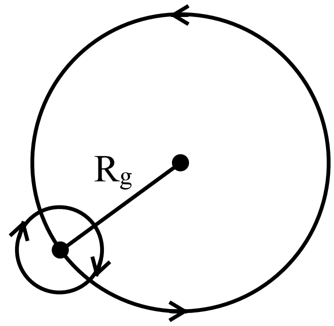
where we use $r = x + R_g$ in the tangential equation. Assuming that $x \ll R_g$ and defining $\Omega_g \equiv \Omega(R_g)$, we can approximate the tangential component to be:

$$\dot{\psi} \approx \Omega_g \left(1 - \frac{2x}{R_g}\right) \ . \tag{153}$$

We then integrate over time, so

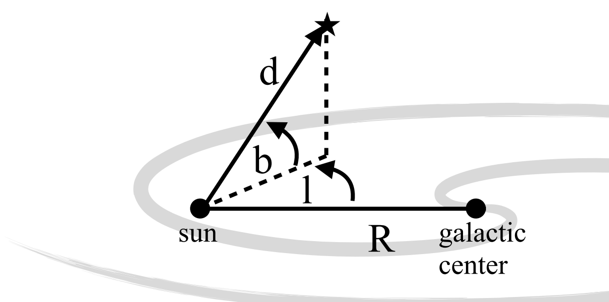
$$\psi(t) = \Omega_g t + \psi_0 - \frac{2\Omega_g r_0}{\kappa R_g}\sin(\kappa t + \alpha)\ . \tag{154}$$

This gives us circular motion of the guiding center with a closed retrograde elliptical orbit in the frame of the guiding center. We also have oscillations in the z direction with frequency ν.



Oort constants: (see Problem Set 3)
Goal: measure the epicycle frequency κ at the position of the sun in the Milky Way by measuring the motion of nearby stars (proper motion an the sky and line of sight velocity).

We use the *galactic coordinate system* to measure the location of stars in the sky (l, b):



l: galactic longitude
b: galactic latitude

R is the distance from the sun to the galactic center ($\sim$ 8kpc) and d is the distance from the sun to the star. l measures the angle in the plane of the Milky Way away from the line of sight to the galactic center, and b measures the angle above the plane of the galaxy. Within this system, we find:

$$\begin{aligned} \text{proper motion}: \mu &\approx d(A\cos(2l) + B) \\ \text{line of sight motion}: v_\parallel &\approx dA\sin(2l) \end{aligned} \tag{155}$$

where A and B are the *Oort constants* given by:

$$\begin{aligned} A &= -\frac{1}{2}\frac{d\Omega}{dR} \\ B &= -\left(\Omega + \frac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\ . \end{aligned} \tag{156}$$

More importantly, they can be related to κ:

$$\boxed{\kappa^2 = -4B(A-B)}\ . \tag{157}$$

Luminosity-velocity relations:
We can relate properties of a galaxy to observables through several equations:

$$\begin{aligned} \theta &= \frac{R}{d} \text{ (apparent size)} \\ F &= \frac{L}{4\pi d^2} \\ v^2 &= \frac{GM}{R} \;. \end{aligned} \tag{158}$$

Introducing surface brightness Σ

$$\begin{aligned} \Sigma &= \frac{F}{\theta^2} = \frac{L}{4\pi d^2} \cdot \frac{d^2}{R^2} \\ &= \frac{L}{4\pi} \cdot \frac{v^4}{G^2 M^2} \end{aligned} \tag{159}$$

then

$$L = \frac{v^4}{\Sigma 4\pi G^2 (M/L)^2} \;. \tag{160}$$

If we assume, for a given class of galaxies, that the surface brightness and the mass-to-light ratio are the same, then

$$\boxed{L \propto v^4} \;. \tag{161}$$

This introduces two important relations.

The *Tully-Fischer relation* is used for spiral galaxies and relates the maximum velocity in the rotation curve v_{max}, which can be measured from HII spectra, and the luminosity:

$$L \propto v_{\text{max}}^4 \;. \tag{162}$$

The *Faber-Jackson relation* is used for ellipticals and relates the velocity dispersion σ_{v} to the luminosity:

$$L \propto \sigma_{\text{v}}^4 \;. \tag{163}$$

Thus, we can get an estimate of the intrinsic luminosity of a galaxy be measuring stellar velocities. The constant of proportionality is roughly $L_*/(220\,\text{km/s})^4$, where L_* is the characteristic galaxy luminosity.

3.C Phase-space distribution function

We have described the individual orbits in a potential, but this is not sufficient to describe galactic dynamics. We want information of the configuration of all particles. Each star is described by its position $\vec{x}$ and velocity $\vec{v}$, and we need to know this for all stars, i.e. how stars are distributed in the 6D phase space $(\vec{x}, \vec{v})$.

We define a *phase-space distribution function*

$$f(\vec{x}, \vec{v}, t) \mathrm{d}^3\, \vec{x}\; \mathrm{d}^3 \vec{v} \tag{164}$$