Luminosity-velocity relations:
We can relate properties of a galaxy to observables through several equations:

$$\begin{aligned} \theta &= \frac{R}{d} \text{ (apparent size)} \\ F &= \frac{L}{4\pi d^2} \\ v^2 &= \frac{GM}{R} \; . \end{aligned} \tag{158}$$

Introducing surface brightness Σ

$$\begin{aligned} \Sigma &= \frac{F}{\theta^2} = \frac{L}{4\pi d^2} \cdot \frac{d^2}{R^2} \\ &= \frac{L}{4\pi} \cdot \frac{v^4}{G^2 M^2} \end{aligned} \tag{159}$$

then

$$L = \frac{v^4}{\Sigma 4\pi G^2 (M/L)^2} \; . \tag{160}$$

If we assume, for a given class of galaxies, that the surface brightness and the mass-to-light ratio are the same, then

$$\boxed{L \propto v^4} \; . \tag{161}$$

This introduces two important relations.

The *Tully-Fischer relation* is used for spiral galaxies and relates the maximum velocity in the rotation curve $v_{\max}$, which can be measured from HII spectra, and the luminosity:

$$L \propto v_{\max}^4 \; . \tag{162}$$

The *Faber-Jackson relation* is used for ellipticals and relates the velocity dispersion σ_{v} to the luminosity:

$$L \propto \sigma_{\mathrm{v}}^4 \; . \tag{163}$$

Thus, we can get an estimate of the intrinsic luminosity of a galaxy be measuring stellar velocities. The constant of proportionality is roughly $L_*/(220\,\mathrm{km/s})^4$, where L_* is the characteristic galaxy luminosity.

3.C Phase-space distribution function

We have described the individual orbits in a potential, but this is not sufficient to describe galactic dynamics. We want information of the configuration of all particles. Each star is described by its position $\vec{x}$ and velocity $\vec{v}$, and we need to know this for all stars, i.e. how stars are distributed in the 6D phase space $(\vec{x}, \vec{v})$.

We define a *phase-space distribution function*

$$f(\vec{x}, \vec{v}, t) \mathrm{d}^3\, \vec{x}\, \mathrm{d}^3 \vec{v} \tag{164}$$

as the probability that at time t, a randomly chosen star has $(\vec{x}_*, \vec{v}_*) \in ([\vec{x}, \vec{x}+d\vec{x}], [\vec{v}, \vec{v}+d\vec{v}])$. This means that the function must be normalized for all t, i.e.

$$\int f(\vec{x}, \vec{v}, t) \mathrm{d}^3 \vec{x} \, \mathrm{d}^3 \vec{v} = 1 \, . \tag{165}$$

Collisionless Boltzmann equation:
We want to describe the time evolution of $f(\vec{x}, \vec{v}, t)$. Since probability cannot be destroyed, the 6D continuity equation must hold.

We define the 6D phase-space vector

$$\vec{w} = (\vec{x}, \vec{v}) \tag{166}$$

then

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{w}} \left(f \dot{\vec{w}} \right) = 0 \, . \tag{167}$$

This is the same form as the standard 3D continuity equation. We can rewrite this by expanding out $\vec{w}$ and using velocity $\vec{v} = \dot{\vec{x}}$ and acceleration $\vec{a} = \dot{\vec{v}}$:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{w}} \left(f \dot{\vec{w}} \right) \\ &= \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{x}} (f \dot{\vec{x}}) + \frac{\partial}{\partial \vec{v}} (f \dot{\vec{v}}) \\ &= \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{x}} (f \vec{v}) + \frac{\partial}{\partial \vec{v}} \left(f (-\vec{\nabla} \phi) \right) \\ &= \frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}} \, . \end{aligned} \tag{168}$$

This gives us the *collisionless Boltzmann equation* (CBE):

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}} = 0} \, . \tag{169}$$

Note that another way to see this is by writing out $\frac{df}{dt} = 0$ and taking the limits $\lim_{\vec{x} \to \infty} = 0$ and $\lim_{\vec{v} \to \infty} = 0$.

General Jeans equations:
A solution to the collisionless Boltzmann equation is difficult to obtain, so we instead study moments of the CBE and the phase-space distribution.

Moments of the phase-space density give us some average quantities of the system.

a) The first moment gives the density n of the system:

$$n = \int f \, \mathrm{d}^3 \vec{v} \, . \tag{170}$$

b) The second moment gives the average velocity $\bar{v}_i$:

$$\bar{v}_i = \frac{1}{n} \int v_i f \, \mathrm{d}^3\vec{v} \, . \tag{171}$$

c) The third moment gives the velocity dispersion σ_{ij}^2:

$$\begin{aligned} \overline{v_i v_j} &= \frac{1}{n} \int v_i v_j f \, \mathrm{d}^3\vec{v} \\ \sigma_{ij}^2 &= \overline{v_i v_j} - \bar{v}_i \bar{v}_j = \overline{(v_i - \bar{v}_i)(v_j - \bar{v}_j)} \, . \end{aligned} \tag{172}$$

We now examine moments of the collisionless Boltzmann equation more closely. We break each integral into three terms to simplify each individually.

a) First moment:

$$\begin{gathered} \int \mathrm{d}^3\vec{v} \left(\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}} \right) = 0 \\ \underbrace{\int \mathrm{d}^3\vec{v} \, \frac{\partial f}{\partial t}}_{①} + \underbrace{\int \mathrm{d}^3\vec{v} \, \vec{v}\frac{\partial f}{\partial \vec{x}}}_{②} - \underbrace{\int \mathrm{d}^3\vec{v} \, \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}}}_{③} = 0 \end{gathered} \tag{173}$$

$$\begin{aligned} ① &: \int \mathrm{d}^3\vec{v} \, \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int \mathrm{d}^3\vec{v} f = \frac{\partial n}{\partial t} \\ ② &: \int \mathrm{d}^3\vec{v} \, \vec{v}\frac{\partial f}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}} \left(\int \mathrm{d}^3\vec{v} \, \vec{v} f \right) = \frac{\partial}{\partial \vec{x}} \left(n \bar{\vec{v}} \right) = \sum_i \frac{\partial}{\partial x_i} \left(n \bar{v}_i \right) \\ ③ &: \int \mathrm{d}^3\vec{v} \, \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}} = \frac{\partial \phi}{\partial \vec{x}} \int \mathrm{d}^3\vec{v} \, \frac{\partial f}{\partial \vec{v}} = \frac{\partial \phi}{\partial \vec{x}} \left[f \right]_{\vec{v}=-\infty}^{\vec{v}=+\infty} = 0 \end{aligned} \tag{174}$$

For the third term, we used the fact that phase-space distribution goes to 0 at $\pm\infty$ for physical systems.

This gives us the *3D continuity equation*:

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(n \bar{\vec{v}} \right) = 0} \, . \tag{175}$$

b) Second moment:

$$\begin{gathered} \int \mathrm{d}^3\vec{v} \, v_j \left(\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}} \right) = 0 \\ \underbrace{\int \mathrm{d}^3\vec{v} \, v_j \frac{\partial f}{\partial t}}_{①} + \underbrace{\int \mathrm{d}^3\vec{v} \, v_j \vec{v}\frac{\partial f}{\partial \vec{x}}}_{②} - \underbrace{\int \mathrm{d}^3\vec{v} \, v_j \frac{\partial \phi}{\partial \vec{x}} \frac{\partial f}{\partial \vec{v}}}_{③} = 0 \end{gathered} \tag{176}$$

$$\textcircled{1}: \int \mathrm{d}^3\vec{v}\, v_j \frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int \mathrm{d}^3\vec{v}\, v_j f = \frac{\partial}{\partial t}\left(n\bar{v}_j\right) = \frac{\partial n}{\partial t}\bar{v}_j + n\frac{\partial \bar{v}_j}{\partial t}$$

$$= -\bar{v}_j \sum_i \frac{\partial}{\partial x_i}\left(n\bar{v}_i\right) + n\frac{\partial \bar{v}_j}{\partial t} = n\frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j \sum_i \frac{\partial}{\partial x_i}\left(n\bar{v}_i\right)$$

using the continuity equation $\frac{\partial n}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(n\bar{v}_i\right)$

to go from the first line to the second

$$\textcircled{2}: \int \mathrm{d}^3\vec{v}\, v_j \vec{v}\frac{\partial f}{\partial \vec{x}} = \int \mathrm{d}^3\vec{v}\, v_j \sum_i v_i \frac{\partial f}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i} \underbrace{\int \mathrm{d}^3\vec{v}\, v_j v_i f}_{= n\overline{v_j v_i} = n\left(\sigma_{ij}^2 + \bar{v}_i\bar{v}_j\right)}$$

$$= \sum_i \frac{\partial}{\partial x_i}\left(n\left(\sigma_{ij}^2 + \bar{v}_i\bar{v}_j\right)\right)$$

$$\textcircled{3}: \int \mathrm{d}^3\vec{v}\, v_j \frac{\partial \phi}{\partial \vec{x}}\frac{\partial f}{\partial \vec{v}} = \int \mathrm{d}^3\vec{v}\, v_j \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = \sum_i \frac{\partial \phi}{\partial x_i}\int \mathrm{d}^3\vec{v}\ v_j \frac{\partial f}{\partial v_i} \tag{177}$$

$((k,l,i)$ are permutations of $(1,2,3))$

$$= \sum_i \frac{\partial \phi}{\partial x_i}\int \mathrm{d}v_k \int \mathrm{d}v_l \underbrace{\int \mathrm{d}v_i \left(v_j \frac{\partial f}{\partial v_i}\right)}$$

$$= \left[v_j f\right]_{v_i=-\infty}^{v_i=+\infty} - \int \mathrm{d}v_i \frac{\partial v_j}{\partial v_i} f$$

$$= 0 - \int \mathrm{d}v_i \delta_{ij} f$$

$$= -\sum_i \frac{\partial \phi}{\partial x_i}\int \mathrm{d}v_k \int \mathrm{d}v_l \int \mathrm{d}v_i \delta_{ij} f$$

$$= -\sum_i \frac{\partial \phi}{\partial x_i}\int \mathrm{d}^3\vec{v}\delta_{ij} f$$

$$= -n\frac{\partial \phi}{\partial x_j}$$

Plugging each term back in, we get

$$n\frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j \sum_i \frac{\partial}{\partial x_i}(n\bar{v}_i) + \sum_i \frac{\partial}{\partial x_i}\left[n\left(\sigma_{ij}^2 + \bar{v}_i\bar{v}_j\right)\right] + n\frac{\partial \phi}{\partial x_i} = 0 \tag{178}$$

which we can rewrite

$$
\begin{aligned}
n\frac{\partial \bar{v}_j}{\partial t} - \underline{\bar{v}_j \sum_i \frac{\partial}{\partial x_i}(n\bar{v}_i)} + \sum_i \frac{\partial}{\partial x_i}(n\sigma_{ij}^2) + \underbrace{\sum_i \frac{\partial}{\partial x_i}(n\bar{v}_i\bar{v}_j)} + n\frac{\partial \phi}{\partial x_j} = 0 \\
= \sum_i (n\bar{v}_i)\frac{\partial}{\partial x_i}\bar{v}_j + \underline{\sum_i \bar{v}_j \frac{\partial}{\partial x_i}(n\bar{v}_i)}
\end{aligned}
\tag{179}
$$

where the two underlined terms cancel. This gives us

$$
n\frac{\partial \bar{v}_j}{\partial t} + \sum_i (n\bar{v}_i)\frac{\partial}{\partial x_i}\bar{v}_j + \sum_i \frac{\partial}{\partial x_i}(n\sigma_{ij}^2) + n\frac{\partial \phi}{\partial x_j} = 0 \; . \tag{180}
$$

This is the *Jeans equation*, often written

$$
\boxed{\frac{\partial \bar{v}_j}{\partial t} + \sum_i \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\frac{1}{n}\sum_i \frac{\partial (n\sigma_{ij}^2)}{\partial x_i} - \frac{\partial \phi}{\partial x_j}} \tag{181}
$$

Each term can be physically interpreted:

$$
\begin{aligned}
\frac{\partial \bar{v}_j}{\partial t} &: \text{acceleration of fluid} \\
\sum_i \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} &: \text{kinematic viscosity/shear} \\
-\frac{1}{n}\sum_i \frac{\partial (n\sigma_{ij}^2)}{\partial x_i} &: \text{pressure} \\
-\frac{\partial \phi}{\partial x_j} &: \text{gravity}
\end{aligned}
\tag{182}
$$

Jeans equations in spherical systems:
We can convert to spherical coordinates and take velocity moments to give us the Jeans equations in spherical coordinates. This is complicated!

To simplify, we take the radial Jeans equation and focus on steady-state symmetric systems.

Implications:

- $\frac{\partial}{\partial t} = 0$ since we have steady state
- $\bar{v}_r = 0$ otherwise we have net radial motion
- $\bar{v}_\theta = \bar{v}_\phi = 0$ or the symmetry is broken
- $\sigma_{r\phi}^2 = \sigma_{r\theta}^2 = 0$ or the symmetry is broken
- $\sigma_{\phi\phi}^2 = \sigma_{\theta\theta}^2 \equiv \sigma_t^2$ or the symmetry is broken.

The simplified Jeans equation is:

$$\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2(\sigma_{rr}^2 - \sigma_t^2)}{r} = -\frac{\partial \phi}{\partial r} = -\frac{GM(<r)}{r^2} \tag{183}$$

where we've plugged in gravity as the force.

We have three limits we can look at:

- $\sigma_{rr}^2 \ll \sigma_t^2$: nearly circular orbits
- $\sigma_{rr}^2 \gg \sigma_t^2$: nearly radial orbits
- $\sigma_{rr}^2 = \sigma_t^2$: isotropic orbits

We define the *anisotropy parameter*:

$$\beta = 1 - \frac{\sigma_t^2}{\sigma_{rr}^2} \tag{184}$$

which gives us a useful form of the Jeans equation for observations:

$$\boxed{\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2\beta\sigma_{rr}^2}{r} = -\frac{GM(<r)}{r^2}}\,. \tag{185}$$

This depends only on radial components with uncertainty from β, assuming spherical symmetry and a steady-state system.

This can be simplified further to get mass estimates:

$$\begin{aligned} M(<r) &= -\frac{r^2}{G}\left(\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2\beta\sigma_{rr}^2}{r}\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{r}{n\sigma_{rr}^2}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + 2\beta\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{r}{n}\frac{\mathrm{d}n}{\mathrm{d}r} + \frac{r}{\sigma_{rr}^2}\frac{\mathrm{d}\sigma_{rr}^2}{\mathrm{d}r} + 2\beta\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln \sigma_{rr}^2}{\mathrm{d}\ln r} + 2\beta\right) \end{aligned} \tag{186}$$

where the last line can be measured with observations.

3.D Stability of stellar systems

The existence of equilibrium solutions to the collisionless Boltzmann equation does not assure stability. Real stellar systems are subject to perturbations. What is important for stability?

MIT OpenCourseWare
https://ocw.mit.edu/

8.902 Astrophysics II
Fall 2023

For information about citing these materials or our Terms of Use, visit:
https://ocw.mit.edu/terms.