The simplified Jeans equation is:

$$\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2(\sigma_{rr}^2 - \sigma_t^2)}{r} = -\frac{\partial \phi}{\partial r} = -\frac{GM(< r)}{r^2} \tag{183}$$

where we've plugged in gravity as the force.

We have three limits we can look at:

- $\sigma_{rr}^2 \ll \sigma_t^2$: nearly circular orbits
- $\sigma_{rr}^2 \gg \sigma_t^2$: nearly radial orbits
- $\sigma_{rr}^2 = \sigma_t^2$: isotropic orbits

We define the *anisotropy parameter*:

$$\beta = 1 - \frac{\sigma_t^2}{\sigma_{rr}^2} \tag{184}$$

which gives us a useful form of the Jeans equation for observations:

$$\boxed{\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2\beta\sigma_{rr}^2}{r} = -\frac{GM(< r)}{r^2}}\,. \tag{185}$$

This depends only on radial components with uncertainty from β, assuming spherical symmetry and a steady-state system.

This can be simplified further to get mass estimates:

$$\begin{aligned} M(< r) &= -\frac{r^2}{G}\left(\frac{1}{n}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + \frac{2\beta\sigma_{rr}^2}{r}\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{r}{n\sigma_{rr}^2}\frac{\partial}{\partial r}(n\sigma_{rr}^2) + 2\beta\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{r}{n}\frac{\mathrm{d}n}{\mathrm{d}r} + \frac{r}{\sigma_{rr}^2}\frac{\mathrm{d}\sigma_{rr}^2}{\mathrm{d}r} + 2\beta\right) \\ &= -\frac{r\sigma_{rr}^2}{G}\left(\frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln \sigma_{rr}^2}{\mathrm{d}\ln r} + 2\beta\right) \end{aligned} \tag{186}$$

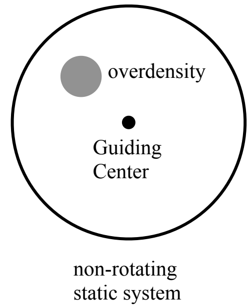
where the last line can be measured with observations.

3.D Stability of stellar systems

The existence of equilibrium solutions to the collisionless Boltzmann equation does not assure stability. Real stellar systems are subject to perturbations. What is important for stability?

Small scales: Jeans instability and random motions

Consider a nearly uniform distribution of stars with perturbations with respect to a static uniform background. We can study the stability of this configuration by inspecting the continuity and the Jeans equations.



We first rewrite and simplify the Jeans equations:

$$\frac{\partial \bar{v}_j}{\partial t} + \sum_i \bar{v}_i \frac{\partial \bar{v} j}{\partial x_i} = -\frac{1}{n} \sum_i \frac{\partial (n\sigma_{ij})}{\partial x_i} - \frac{\partial \phi}{\partial x_j} \ . \tag{187}$$

We can rewrite the number density n using $\rho = mn$ and assume that σ_{ij} is isotropic so the pressure is $P = \rho\sigma_{ij}^2 = \rho\sigma_{ij} = mn\sigma_{ij}^2$. Then we can rewrite the Jeans equations as:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla}\phi - \frac{1}{\rho}\vec{\nabla} P \ . \tag{188}$$

Similarly, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \ . \tag{189}$$

Note that we have dropped the $^-$'s (average value symbols) for simplicity in our equations and $\vec{v}$ is referring to the average velocities at $(\vec{x}, t)$. We will continue with this convention in the following calculations.

Small perturbations:
For a small perturbation in a static uniform background, we have

$$\begin{aligned} \rho &= \rho_0 + \epsilon\rho_1(\vec{x}, t) \\ \vec{v} &= \vec{v}_0 + \epsilon\vec{v}_1(\vec{x}, t) \\ P &= P_0 + \epsilon P_1(\vec{x}, t) \\ \phi &= \phi_0 + \epsilon\phi_1(\vec{x}, t) \ . \end{aligned} \tag{190}$$

We can choose $\phi_0 = 0$ and, since the background is static, $\vec{v}_0 = \vec{0}$. ρ_0 and P_0 are both nonzero constants. Note that this is not a physical set of conditions since Poisson's equation gives $\nabla^2\phi_0 = 4\pi G\rho_0$ so $\phi_0 = 0$ implies $\rho_0 = 0$, but we continue with our calculations ignoring this. This is known as the *Jeans swindle*.

We can plug this into the continuity equation:

$$\frac{\partial}{\partial t}\rho_0 + \epsilon\frac{\partial}{\partial t}\rho_1 + \vec{\nabla}\left(\rho_0\vec{v}_0 + \epsilon\rho_1\vec{v}_0 + \epsilon\rho_0\vec{v}_1 + \epsilon^2\rho_1\vec{v}_1\right) = 0 \ . \tag{191}$$

Performing derivatives on constants and neglecting terms of order ϵ^2, this becomes:

$$\begin{aligned}
&\epsilon\frac{\partial}{\partial t}\rho_1 + \vec{\nabla}\cdot(\epsilon\rho_0\vec{v}_1) = 0 \\
\Rightarrow &\frac{\partial\rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v}_1 = 0\ .
\end{aligned} \tag{192}$$

We then plug this into the Jeans equation:

$$\left(\frac{\partial\vec{v}_0}{\partial t} + \epsilon\frac{\partial\vec{v}_1}{\partial t}\right) + \left((\vec{v}_0 + \epsilon\vec{v}_1)\cdot\vec{\nabla}\right)(\vec{v}_0 + \epsilon\vec{v}_1) = -\vec{\nabla}(\phi_0 + \epsilon\phi_1) - \frac{1}{\rho_0 + \epsilon\rho_1}\vec{\nabla}(P_0 + \epsilon P_1) \tag{193}$$

then

$$\epsilon\frac{\partial\vec{v}_1}{\partial t} = -\epsilon\vec{\nabla}\phi_1 - \underbrace{\frac{1}{\rho_0 + \epsilon\rho_1}\vec{\nabla}(\epsilon P_1)}_{\approx\,\epsilon\frac{\vec{\nabla}P_1}{\rho_0}}\ . \tag{194}$$

We can write

$$\vec{\nabla}P_1 = \frac{\partial P_1}{\partial\vec{x}} = \underbrace{\left(\frac{\partial P_1}{\partial\rho}\right)\Big|_{\rho_0}}_{v_s^2}\frac{\partial\rho_1}{\partial\vec{x}} \tag{195}$$

where v_s is the sound speed, or the speed at which perturbations can propagate. Returning to the previous equation, this gives us:

$$\begin{aligned}
&\epsilon\frac{\partial\vec{v}_1}{\partial t} = -\epsilon\vec{\nabla}\phi_1 - \epsilon\frac{v_s^2}{\rho_0}\vec{\nabla}\rho_1 \\
\Rightarrow &\frac{\partial\vec{v}_1}{\partial t} = -\vec{\nabla}\phi_1 - \frac{v_s^2}{\rho_0}\vec{\nabla}\rho_1\ .
\end{aligned} \tag{196}$$

We now combine the time derivative of the continuity with the Jeans equation:

$$\begin{aligned}
&\frac{\partial^2\rho_1}{\partial t^2} + \frac{\partial}{\partial t}\left(\rho_0\vec{\nabla}\cdot\vec{v}_1\right) = 0 \\
\Rightarrow &\frac{\partial^2\rho_1}{\partial t^2} + \rho_0\underbrace{\vec{\nabla}\cdot\left(\frac{\partial\vec{v}_1}{\partial t}\right)}_{=-\vec{\nabla}^2\phi_1 - \frac{v_s^2}{\rho_0}\vec{\nabla}^2\rho_1} = 0
\end{aligned} \tag{197}$$

so

$$\frac{\partial^2\rho_1}{\partial t^2} + \rho_0\left(-\vec{\nabla}^2\phi_1 - \frac{v_s^2}{\rho_0}\underbrace{\vec{\nabla}^2\rho_1}\right) = 0\ . \tag{198}$$
$$= 4\pi G\rho_1 \text{ (from Poisson's Equation)}$$

Finally, we get a wave equation for ρ_1:

$$\boxed{\frac{\partial^2\rho_1}{\partial t^2} - 4\pi G\rho_0\rho_1 - v_s^2\vec{\nabla}^2\rho_1 = 0}\ . \tag{199}$$

We use an ansatz for the solution of the form

$$\rho_1 = C \exp^{i(\vec{k}\cdot\vec{c} - \omega t)} \tag{200}$$

which gives the time evolution of perturbations. We plug this into the wave equation and get

$$w^2 = v_s^2 k^2 - 4\pi G \rho_0 \,. \tag{201}$$

We have two solutions:

$w^2 > 0$: the exponent is imaginary, so we get stable oscillating modes

$w^2 < 0$: the exponent is real, so we get unstable growing or decaying modes

If $w = 0$:

$$\lambda_J^2 = \left(\frac{2\pi}{k}\right)^2 = \frac{\pi v_s^2}{G\rho_0} \,. \tag{202}$$

Jeans length and mass:
The *Jeans length* λ_J is the maximum size a perturbation can be to remain stable. The *Jeans mass* M_J is the corresponding mass enclosed within the Jeans length of a given substance.

$$\boxed{\begin{aligned} \lambda_J^2 &= \frac{\pi v_s^2}{G\rho_0} = \frac{\pi\sigma^2}{G\rho_0} \\ M_J &= \frac{4}{3}\pi\rho_0\lambda_J^3 \end{aligned}} \,. \tag{203}$$

So we have stability for $\lambda < \lambda_J$ and $M < M_J$. Note that for collisional gas, the Jeans length is determined by the sound speed v_s and for collisionless dark matter and stars, the Jeans length is determined by the pressure from the velocity dispersion σ.

Meaning of the Jeans length:
If perturbations can be crossed before collapse, pressure can stabilize the collapse.

The free-fall time is

$$t_{\text{ff}} \sim \frac{1}{\sqrt{G\rho}} \tag{204}$$

and the perturbation crossing time is

$$t_{\text{cross}} \sim \frac{r}{v_s} \,. \tag{205}$$

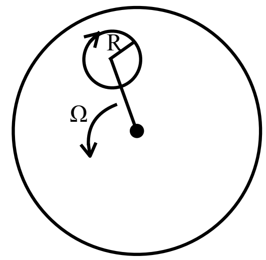
Then we get collapse if

$$\begin{aligned} t_{\text{cross}} &> t_{\text{ff}} \\ \frac{r}{v_s} &> \frac{1}{\sqrt{G\rho}} \\ \Rightarrow r^2 &> \frac{v_s^2}{G\rho} \end{aligned} \tag{206}$$

which is similar to the Jeans length result, differing only by a factor of π. So, random motion and pressure can stabilize perturbations on small scales.

Large scales: Toomre instability and rotational motion.

Consider a rotating stellar disk where radial perturbations can occur. We study the stability of this configuration by inspecting the centripetal and acceleration forces. Note that mass and angular momentum are conserved during the perturbation: $\dot{m} = \dot{L} = 0$.



rotating
non-static system

During the perturbation, $R \to R'$ with $R' = R - \mathrm{d}r$. We want to know when this will lead to collapse and when it will be stable. This is a competition between centripetal and gravitational forces.

The change in gravitational acceleration is

$$\begin{aligned} a_g &= \frac{G\pi R^2\Sigma}{R'^2}, \text{ and } \pi R^2\Sigma \text{ is mass } (\Sigma \text{ is surface density}) \\ \Rightarrow \frac{da_g}{dR'} &= \frac{-2G\pi R^2\Sigma}{R'^3} \,. \end{aligned} \tag{207}$$

The change in centripetal acceleration, with rotational frequency of the patch Ω, is

$$\begin{aligned} L &= \Omega R^2 = \Omega' R'^2 \text{ (since } \dot{L} = 0) \\ \Rightarrow \Omega' &= \Omega \left(\frac{R}{R'}\right)^2 \,. \end{aligned} \tag{208}$$

So

$$\begin{aligned} a_c &= \frac{R'^2\Omega'^2}{R'} = R'\Omega'^2 = \Omega^2 \frac{R^4}{R'^3} \\ \Rightarrow \frac{\mathrm{d}a_c}{\mathrm{d}R'} &= \frac{-3\Omega^2 R^4}{R'^4} \,. \end{aligned} \tag{209}$$

Stability: the system is stable if $|\mathrm{d}a_g| < |\mathrm{d}a_c|$. So we need

$$\begin{aligned} \frac{2\pi G R^2\Sigma}{R'^3} &< \frac{3\Omega^2 R^4}{R'^4} \\ \Rightarrow \frac{2\pi G\Sigma}{3\Omega^2} &< R \underbrace{\frac{R}{R'}}_{\approx 1} \end{aligned} \tag{210}$$

and the disk is stable if

$$\boxed{R_{\mathrm{rot}} > \frac{2\pi G\Sigma}{3\Omega^3}} \,. \tag{211}$$

Full stability criterion: On small scales, we have stability if $R < \lambda_J$ and on large scales, we have stability if $R > R_{\text{rot}}$. Small scales are stabilized by random motion and large scales are stabilized by rotational motion. The system is unstable if $\lambda_J < R < R_{\text{rot}}$. We can combine the two criteria and get full stability when $\lambda_J \geq R_{\text{rot}}$. This gives us (adapting λ_J from an arbitrary 3D potential to a 2D disk):

$$\begin{aligned} \frac{\pi}{8}\frac{\sigma^2}{G\Sigma} &\geq \frac{2\pi G\Sigma}{3\Omega^2} \\ \Rightarrow \sigma_{\text{crit}} &\geq \frac{4}{\sqrt{3}}\frac{G\Sigma}{\Omega}\,. \end{aligned} \tag{212}$$

Note that the angular speed of the patch is only approximately Ω. It actually rotates with epicyclic frequency κ, which is not too far off from Ω for real galaxies. We can relate κ to Ω for a typical galactic disk:

$$\kappa^2(R_g) = \left(r\frac{\mathrm{d}\Omega^2}{\mathrm{d}r} + 4\Omega^2\right)\Big|_{R_g} \tag{213}$$

and in galaxies with circular velocity that is approximately constant:

$$\Omega = \frac{v_c}{r} \Rightarrow \kappa^2 = 2\Omega^2 \Rightarrow \kappa = \sqrt{2}\Omega\,. \tag{214}$$

So for galaxies:

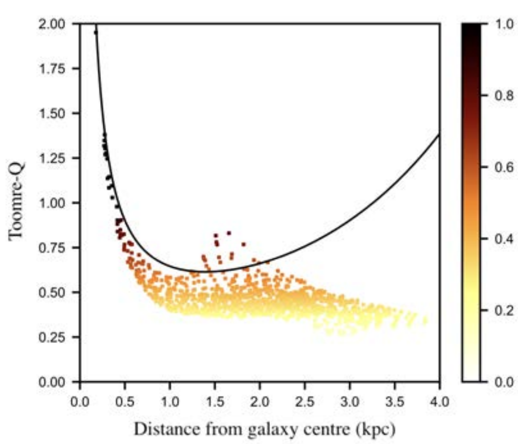
$$\sigma_{\text{crit}} \geq \frac{4}{\sqrt{3}}\frac{G\Sigma}{\kappa^2/\sqrt{2}} = \underbrace{\sqrt{\frac{32}{3}}}_{3.26}\frac{G\Sigma}{\kappa}\,. \tag{215}$$

Toomre finds

$$\sigma_{\text{crit}} = 3.26\frac{G\Sigma}{\kappa} \tag{216}$$

Toomre criterion Q:
We can write the stability criterion Q for rotating disks:

$$\boxed{Q = \frac{\sigma}{\sigma_{\text{crit}}}\begin{cases} > 1 : \text{stable} \\ < 1 : \text{unstable} \end{cases}} \tag{217}$$



Here we show Q as a function of radius from the galactic center for the galaxy DLA0817 (the Wolfe Disk) from Neelemen et al. 2020. The solid line shows Q assuming the gas density falls off exponentially. The points show observed data, which underestimates Q likely due to beam smearing which increases measured surface density.

Extended Data Figure 7b. Neeleman, M., Prochaska, J.X., Kanekar, N. et al. A cold, massive, rotating disk galaxy 1.5 billion years after the Big Bang. Nature 581, 269–272 (2020). https://doi.org/10.1038/s41586-020-2276-y

Code used to generate kinematic models: mneeleman, and J. Xavier Prochaska. “Mneeleman/qubefit: Small Documentation Updates”. Zenodo, February 11, 2021. https://doi.org/10.5281/zenodo.4534407.