Part II

Cosmology and Structure Formation

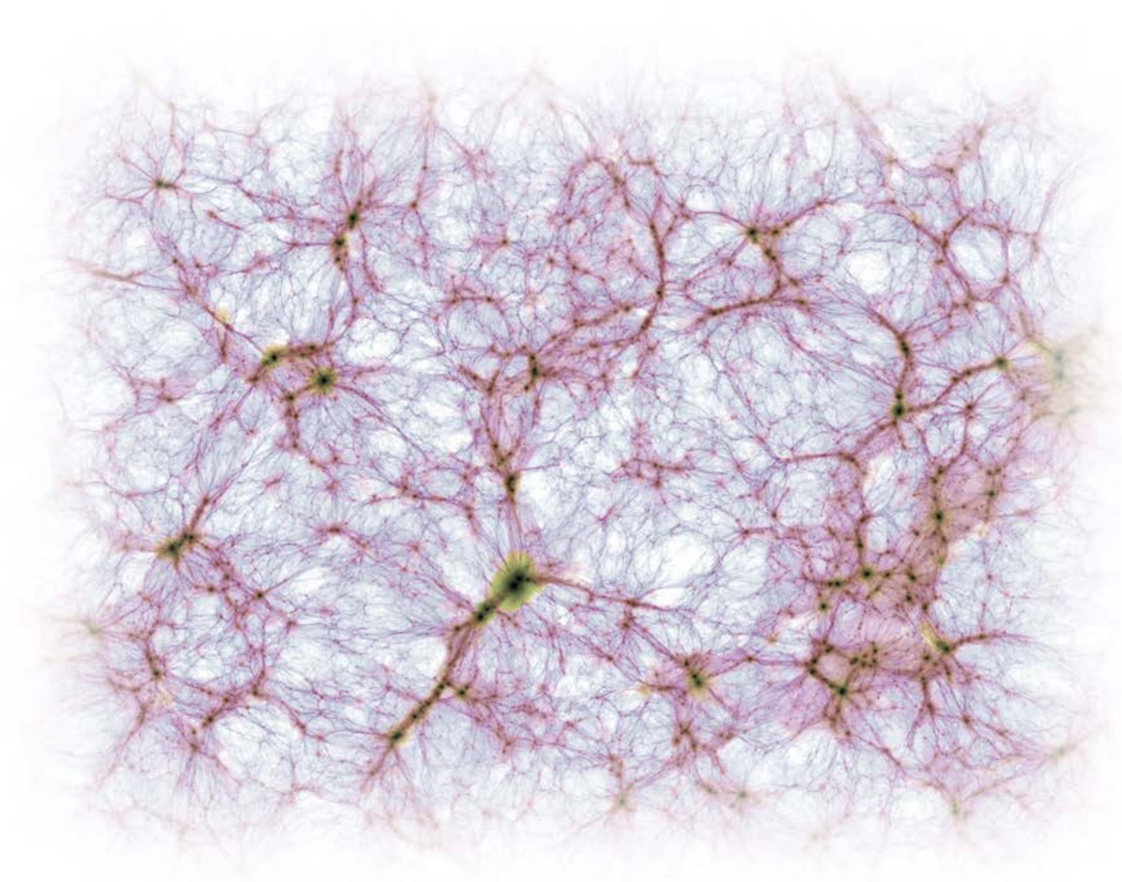


Image: TNG Simulations

1 Cosmology

Cosmology is the study of dynamics of the entire Universe as a single dynamical system.

1.A Cosmological Principle and dynamics

- The Universe is homogeneous: it is uniform on large scales.
- The Universe is isotropic: it looks the same for all observers on large scales.

This implies that the space-time metric is the same everywhere, which generates symmetries and simplifies the solutions to general relativity equations.

Hubble Law:
We observe that

$$\vec{v} = H_0 \vec{r} \tag{267}$$

where $H_0 \sim 70\,\mathrm{km/s/Mpc}$ refers to the present-day Hubble factor. The specific form of this law can be derived from the cosmological principles:

- Linearity (follows from isotropy):
  Suppose $\vec{v} = f(\vec{r})$.
  Then from Observer A's perspective

  $$\vec{v}_1 = f(\vec{r}_1) \text{ and } \vec{v}_2 = f(\vec{r}_2)$$

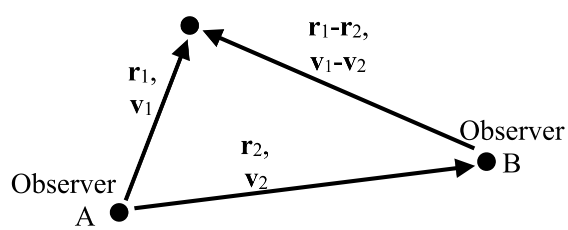
  $$\text{and } \vec{v}_1 - \vec{v}_2 = f(\vec{r}_1) - f(\vec{r}_2)\ .$$

  From Observer B's perspective

  $$\vec{v}_1 - \vec{v}_2 = f(\vec{r}_1 - \vec{r}_2)$$

  so we find that

  $$f(\vec{r}_1 - \vec{r}_2) = f(\vec{r}_1) - f(\vec{r}_2)\ .$$

  This implies that f is linear since isotropy requires that each observer sees the same Hubble law.

- Uniqueness (follows from homogeneity):
  $f(...)$ is linear, so

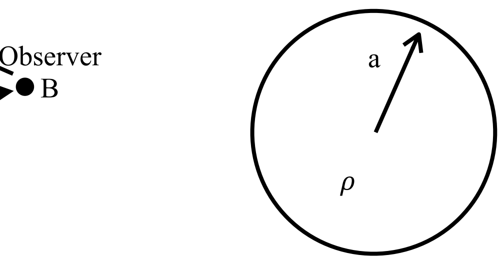
  $$f(\vec{r}) = \underline{\underline{\mathbf{H}}}\vec{r} \tag{268}$$

  where $\underline{\underline{\mathbf{H}}}$ is a matrix.
  We assume that it is non-diagonal, otherwise a special direction would be preferred (since it introduces an axis), and $\underline{\underline{\mathbf{H}}} = H_0\mathbb{1}$. Then

  $$f(\vec{r}) = H_0\vec{r}\,. \tag{269}$$

Dynamics of cosmological expansion:

unsion. General relativity allows a detailed derivation
off's theorem to get initial insight. Birkhoff's theorem
expanding self-gravitating sphere is equivalent to a



$$\begin{aligned} M(<a) &= \frac{4}{3}\pi a^3 \rho \\ \Rightarrow \ddot{a} &= -\frac{GM(<a)}{a^2} = -\frac{4\pi G\rho}{a^2} \cdot \frac{1}{3} a^3 \end{aligned} \tag{270}$$

We multiply each side by $\dot{a}$:

$$\begin{aligned} \ddot{a}\dot{a} &= -\frac{4\pi G\rho}{3} a\dot{a} \\ &= -\frac{4\pi G\rho_0}{3} a_0^3 a^{-2}\dot{a} \;\; (\text{since } \rho = \rho_0 \frac{\mathrm{a}_0^3}{\mathrm{a}^3}) \; . \end{aligned} \tag{271}$$

Since $\ddot{a}\dot{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{a}^2\right)$ and $a^{-2}\dot{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{a}\right)$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{a}^2\right) = -\frac{4\pi G\rho_0 a_0^3}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{a}\right) \; . \tag{272}$$

Integrate $\int dt$ on both sides:

$$\frac{1}{2}\dot{a}^2 = \frac{4\pi G\rho_0 a_0^3}{3}\frac{1}{a} + \tilde{\kappa} \; . \tag{273}$$

Here, $\tilde{\kappa}$ is the integration constant, and we can use the density expression $\rho_0 a_0^3 = \rho a^3$ to simplify our equation:

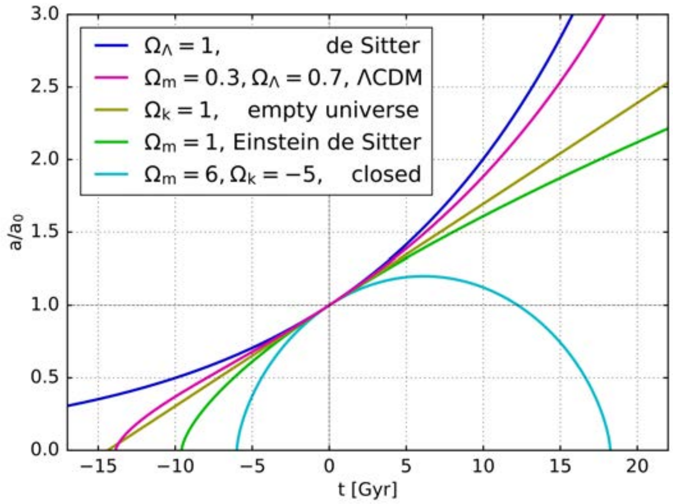
$$\frac{1}{2}\dot{a}^2 = \frac{4\pi G}{3}\rho a^2 + \tilde{\kappa} \; . \tag{274}$$

So the dynamics is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\tilde{k}}{a^2} \; . \tag{275}$$

This is the Friedmann equation for $\Lambda = 0$. With a cosmological constant Λ, we have

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\tilde{k}}{a^2} + \frac{\Lambda}{3}} \; . \tag{276}$$



Here, a is the scale factor such that length $l = l_0 \frac{a}{a_0}$. Also note that:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\dot{r}}{r} = \frac{v}{r} = H \quad \left(H(t) \equiv \frac{\dot{a}}{a}\right)$$

$\dfrac{8\pi G}{3}\rho$ is the matter/radiation density

$\dfrac{\tilde{\kappa}}{a^2}$ is the curvature

$\dfrac{\Lambda}{3}$ is the cosmological constant

Since volume grows with the length cubed and the total mass in the universe is constant, the matter density is proportional to a^{-3}. The radiation density also decreases due to the increasing volume but also decreases as the wavelengths are stretched, so radiation density is proportional to a^{-4}. The dark energy Λ is constant.

The plot above shows how the scale factor grows with time for several different types of universes. Our current understanding of our universe is that it is described by the ΛCDM model, where roughly 30% of the energy budget is matter, 70% is dark energy, and there is a very small amount of radiation and no curvature. If there were positive or negative curvature, we would get an open ore closed universe. There are also several toy universes that are often useful to think about. A flat, dark energy-only universe is the de Sitter model and a flat, matter-only universe is the Einstein-de Sitter model An empty universe has only a curvature term, and is an open universe.

Dynamical evolution of the Universe:
Different terms in the Friedmann equation dominate at different times.

- radiation term $\propto a^{-4} \Longrightarrow$ dominates at very early times
- matter term $\propto a^{-3} \Longrightarrow$ dominates at early times
- curvature term $\propto a^{-2} \Longrightarrow$ dominates at medium times
- Λ term $\propto$ constant $\Longrightarrow$ dominates at late times

Therefore, from the Friedmann equation, we can derive different regimes of the Universe:

- radiation regime

$$\begin{array}{lcl} \dot{a}^2 \propto a^{-2} & & \\ \dot{a} \propto a^{-1} & \Longrightarrow & a \propto t^{\frac{1}{2}} \qquad \Longrightarrow \qquad t_0 = \dfrac{1}{2}\dfrac{1}{H_0} \\ a\mathrm{d}a \propto \mathrm{d}t & & H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} \end{array}$$

- matter regime

$$\begin{array}{l} \dot{a}^2 \propto a^{-1} \\ \dot{a} \propto a^{-\frac{1}{2}} \\ \sqrt{a}\mathrm{d}a \propto \mathrm{d}t \end{array} \implies \begin{array}{l} a \propto t^{\frac{2}{3}} \\ H(t) = \frac{\dot{a}}{a} = \frac{2}{3t} \end{array} \implies t_0 = \frac{2}{3}\frac{1}{H_0}$$

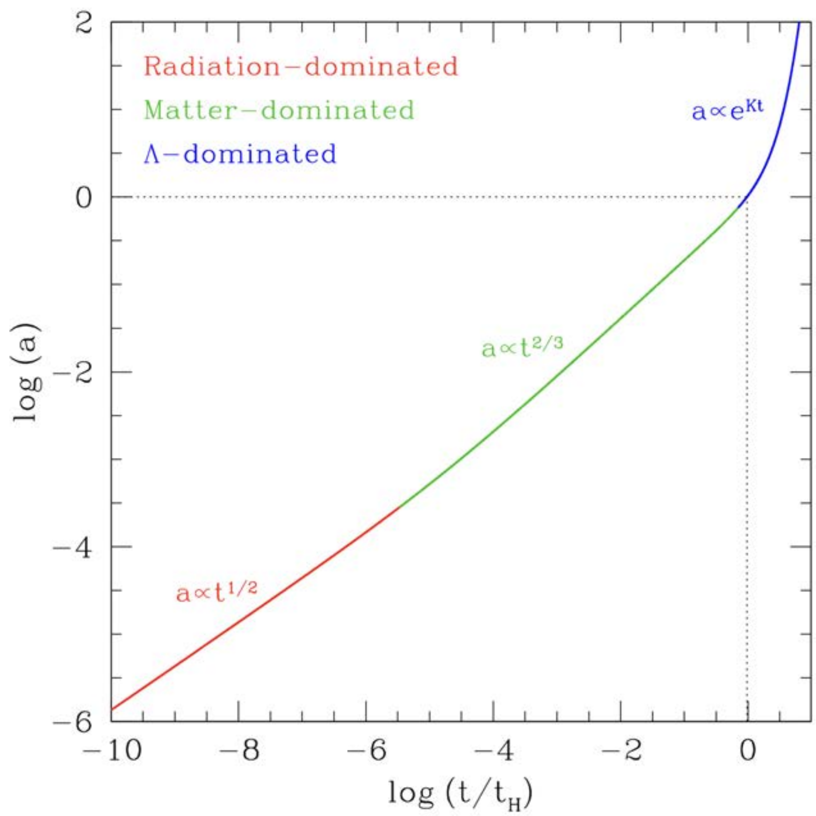
- curvature regime

$$\begin{array}{l} \dot{a}^2 \propto \text{constant} \\ \dot{a} \propto \text{constant} \\ \mathrm{d}a \propto \mathrm{d}t \end{array} \implies \begin{array}{l} a \propto t \\ H(t) = \frac{\dot{a}}{a} = \frac{1}{t} \end{array} \implies t_0 = \frac{1}{H_0}$$

- Λ regime

$$\begin{array}{l} \dot{a}^2 \propto a^2\frac{\Lambda}{3} \\ \dot{a} \propto a\sqrt{\frac{\Lambda}{3}} \\ \frac{\mathrm{d}a}{a} \propto \sqrt{\frac{\Lambda}{3}}\mathrm{d}t \end{array} \implies a \propto e^{\sqrt{\frac{\Lambda}{3}}t} \implies \text{exponential growth}$$

As the universe evolves, it expands at different rates depending on the regime (radiation/matter/Λ).



1.B Dynamics derived with general relativity

Goal: use the field equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{277}$$

to derive the Friedmann equation.

$G_{\mu\nu}$: Einstein tensor; 1$^{\text{st}}$ and 2$^{\text{nd}}$ derivatives of the metric

$T_{\mu\nu}$: stress-energy tensor
$g_{\mu\nu}$: metric (similar to Poisson's equation $\nabla^2\Phi = 4\pi G\rho + \frac{\Lambda}{3}$ with Φ replaced with curvature)

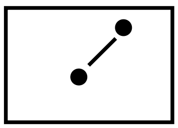
First we need to specify $g_{\mu\nu}$ and $T_{\mu\nu}$.

Metrics:
The space-time interval is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ . \tag{278}$$

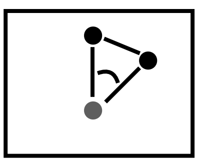
Some examples of spatial metrics:

- 2D-flat space in Cartesian coordinates:



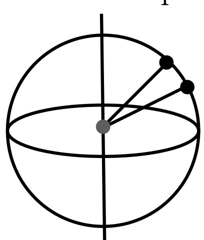
$$\mathrm{d}s^2 = \begin{pmatrix}\mathrm{d}x & \mathrm{d}y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{279}$$

- 2D-flat space polar coordinates:



$$\mathrm{d}s^2 = \begin{pmatrix}\mathrm{d}r & \mathrm{d}\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}\begin{pmatrix}\mathrm{d}r\\ \mathrm{d}\theta\end{pmatrix} = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \tag{280}$$

- 2D-curved space:



$$\mathrm{d}s^2 = \begin{pmatrix}\mathrm{d}\theta & \mathrm{d}\varphi\end{pmatrix}\begin{pmatrix}R^2 & 0\\ 0 & R^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\mathrm{d}\theta\\ \mathrm{d}\varphi\end{pmatrix} = R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi) \tag{281}$$

  We can rewrite $\chi = R\theta$, so

$$\mathrm{d}s^2 = \mathrm{d}\chi^2 + R^2\sin^2\frac{\chi}{R}\mathrm{d}\varphi^2 \ . \tag{282}$$

  When R goes to infinity, we have

$$\sin\left(\frac{\chi}{R}\right) \approx \frac{\chi}{R} \tag{283}$$

$$\Rightarrow \mathrm{d}s^2 = \mathrm{d}\chi^2 + \chi^2\mathrm{d}\varphi^2 \tag{284}$$

  which is gives us flat space!

- 4D space time:

$$g_{\mu\nu} = \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{285}$$

  and

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ . \tag{286}$$

Robertson-Walker metric:
The metric form follows from homogeneity and isotropy, and the field equations give us the time evolution:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)\left[\mathrm{d}\chi^2 + f_k^2(\chi)\right]\left(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\varphi^2\right) \tag{287}$$

and

$$f_k(\chi) = \begin{cases} k^{-1/2}\sin\left(k^{1/2}\chi\right), & \text{closed k} > 0 \\ \chi, & \text{flat k} = 0 \\ |k|^{-1/2}\sinh\left(|k|^{1/2}\chi\right), & \text{open k} < 0 \end{cases} \tag{288}$$

with the units for k: $[k] = \frac{1}{L^2}$.

Derivation of the Friedmann equations:
We have the stress energy tensor:

$$T_{\mu\nu} = \begin{pmatrix} T_{00} \cong \text{energy density} & T_{0j} \cong \text{energy flux} \\ T_{j0} \cong \text{momentum density} & T_{ik} \cong \text{stress tensor} \end{pmatrix} \tag{289}$$

The stress tensor T_{ik} is force per unit area:

$$\begin{pmatrix} T_{ii} \cong \text{pressure} & T_{ik} \cong \text{shear} \end{pmatrix} \tag{290}$$

$T_{\mu\nu}$ has to be a perfect fluid with no shear or isotropic pressure:

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu - \frac{p}{c^2}g_{\mu\nu} \tag{291}$$

In the rest frame of a comoving observer:

$$T_{\mu\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{292}$$

To evaluate $G_{\mu\nu}$, we take the derivative of the metric. We then plug this into the Einstein field equations, which gives two independent equations. This leads to the Friedmann equations:

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \end{aligned}}. \tag{293}$$

For relativistic bosons and fermions $p = \rho c^2/3$, and for non-relativistic particles $p = 0$.

Critical density and density parameters:
The *critical density* ρ_{crit} is the density that gives a flat universe ($k = 0$) and is given by

$$\rho_{\text{crit}} = \frac{3H^2(t)}{8\pi G} \tag{294}$$

with present-day value

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \approx 1.8 \times 10^{-29} h^2 \text{ g/cm}^3 \, . \tag{295}$$

For a sphere with radius a filled with the critical density, the gravitational potential is equal to the specific kinetic energy:

$$\frac{G\frac{4}{3}\pi\rho_{\text{crit}}a^3}{a} = \frac{\dot{a}^2}{2} \, . \tag{296}$$

This is the limiting case between an open and closed universe and leads to eternal expansion.

We define the *cosmological density parameters* in terms of the critical density:

$$\begin{aligned}
\Omega_m(t) &= \frac{\rho_m(t)}{\rho_{\text{crit}}(t)} \\
\Omega_r(t) &= \frac{\rho_r(t)}{\rho_{\text{crit}}(t)} \\
\Omega_k(t) &= -\frac{kc^2}{H^2} \\
\Omega_\Lambda(t) &= \frac{\Lambda c^2}{3H^2} = \frac{\rho_\Lambda(t)}{\rho_{\text{crit}}(t)}, \;\; \rho_\Lambda(t) = \frac{\Lambda c^2}{8\pi G} \\
\Omega(t) &= \Omega_m(t) + \Omega_r(t)
\end{aligned} \tag{297}$$

The present-day values are:

$$\begin{aligned}
\Omega_{m,0} &= \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \\
\Omega_{r,0} &= \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \\
\Omega_{k,0} &= -\frac{kc^2}{H_0^2} \\
\Omega_{\Lambda,0} &= \frac{\rho_{\Lambda,0}}{\rho_{\text{crit},0}} \\
\Omega_0 &= \frac{\rho_0}{\rho_{\text{crit},0}}
\end{aligned} \tag{298}$$

so

$$\begin{aligned}
\rho_m &= \Omega_{m,0}\rho_{\text{crit},0}a^{-3} \\
\rho_r &= \Omega_{r,0}\rho_{\text{crit},0}a^{-4} \\
\rho_k &= \Omega_{k,0}\rho_{\text{crit},0}a^{-2} \\
\rho_\Lambda &= \Omega_{\Lambda,0}\rho_{\text{crit},0} \, .
\end{aligned} \tag{299}$$

We also often consider the baryon density parameter Ω_b separately from the total matter density, so the total matter density is the sum of the baryon and dark matter densities $\Omega_m = \Omega_{\text{dm}} + \Omega_b$.

Our current measurements of these values are (from the Planck 2018 results)

$$\begin{aligned}\Omega_{m,0} &= 0.315 \pm 0.007 \\ \Omega_{\mathrm{dm},0} &= 0.264 \pm 0.003 \\ \Omega_{b,0} &= 0.0493 \pm 0.0003 \\ \Omega_{k,0} &= 0.0007 \pm 0.0019 \\ \Omega_{\Lambda,0} &= 0.6847 \pm 0.0073\end{aligned} \tag{300}$$

with $H_0 = 67.4 \pm 0.5\,\mathrm{km/s/Mpc}$. The radiation parameter $\Omega_{r,0}$ can be derived from the measured temperature of the CMB and relating the photon and neutrino density to get $\Omega_{r,0} \approx 10^{-4}$. We discuss how to obtain these values from observations in Part III.

We can rewrite first Friedmann equation:

$$\begin{aligned}H^2(t) &= \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{kc^2}{a^2} \\ &= \frac{8\pi G}{3}\rho_{\mathrm{crit},0}[\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0}] - \frac{kc^2}{a^2} \\ (\rho_{\mathrm{crit},0} &= \frac{3{H_0}^2}{8\pi G}) \\ &= {H_0}^2\left[\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} - \frac{kc^2}{a^2{H_0}^2}\right] \\ (-\frac{kc^2}{{H_0}^2} &= \Omega_{k,0} = 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}) \\ &= {H_0}^2\left[\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}\right]\end{aligned} \tag{301}$$

So we find:

$$\boxed{\begin{aligned}&H^2(a) = {H_0}^2E^2(a) \\ &E^2(a) = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}\end{aligned}} \tag{302}$$

which is a useful form of the Friedmann equation.

Notes:

- Radiation dominates in early times, then matter, then the cosmological constant.
- Matter-Λ equality occurs when $\Omega_\Lambda = \Omega_m$:

$$\begin{aligned}\Omega_{\Lambda,0} &= \frac{\Omega_{m,0}}{a^3} \\ \Longrightarrow a &\approx \frac{1}{2.3}, z \approx 1.3 \ (z \approx 1 \text{ is } 6-7\,\mathrm{Gyr} \text{ after the Big Bang})\ .\end{aligned} \tag{303}$$

- Matter-radiation equality occurs when $\Omega_r = \Omega_m$:

$$\Omega_{r,0}a^{-4} = \Omega_{m,0}a^{-3} \tag{304}$$

$$\Longrightarrow a \approx \frac{1}{3700},\ z = 3700\ . \tag{305}$$

- Observationally, $\Omega_{k,0} \approx 0$.