where dA is area and dr is depth. So

$$\mathrm{d}V_\chi = \underbrace{\underbrace{(D_A{}^2\mathrm{d}\Omega)}_{\text{proper area}}(1+z)^2}_{\text{comoving area}} \cdot \underbrace{\mathrm{d}\chi}_{\text{comoving depth}} \tag{338}$$

$$A_{\text{proper}} = a^2 A_{\text{comoving}} \tag{339}$$

$$\Rightarrow A_{\text{comoving}} = \frac{1}{a^2}A_{\text{proper}} = (1+z)^2 A_{\text{proper}} \tag{340}$$

$\mathrm{d}\chi = \frac{c}{H_0}\frac{\mathrm{d}z}{E(z)}$, so

$$\mathrm{d}V_\chi = (D_A{}^2\mathrm{d}\Omega)(1+z)^2\frac{c}{H_0}\frac{1}{E(z)}dz \tag{341}$$

and finally:

$$\mathrm{d}V_\chi = \frac{c}{H_0}\frac{(1+z)^2 D_A{}^2}{E(z)}\mathrm{d}\Omega\mathrm{d}z\ . \tag{342}$$

Plug in $D_A = \frac{1}{1+z}f_\kappa(\chi)$:

$$\mathrm{d}V_\chi = \frac{c}{H_0}\frac{f_\kappa^2(\chi)}{E(z)}\mathrm{d}z\mathrm{d}\Omega \tag{343}$$

$$= f_k^2(\chi)rd\Omega\frac{\mathrm{d}\chi}{\mathrm{d}z}\mathrm{d}z\ . \tag{344}$$

1.D Inflation

So far, dynamics have been described by the Friedmann equations with some mass-energy content of the Universe: $\Omega_m, \Omega_r, \Omega_k, \Omega_\Lambda$. Is this sufficient to explain all data?

Problems:

- Horizon problem:

  $$\rho_r \propto (1+z)^4 \text{ and } \rho_r \propto T^4 \Rightarrow T \propto (1+z) \tag{345}$$

  The Universe cools and at some z_{recomb}, it consists of neutral hydrogen atoms (recombination). We get the balancing equation

  $$H^+ + e \rightleftharpoons H^0 + \chi \tag{346}$$

  where $\chi = 13.6$ eV is the ionization energy. We also have:

  $$\begin{aligned} x &= \frac{\text{number density of free e}^-}{\text{number density of protons}} \\ \eta &= \frac{n_b}{n_\gamma} = \frac{\text{baryon number density}}{\text{photon number density}} \approx 5\times 10^{-10}\left(\frac{\Omega_{b,0}h^2}{0.01}\right) \end{aligned} \tag{347}$$

which we use in the Saha equation:

$$\frac{1-x}{x^2} \approx 3.84\eta \left(\frac{k_B T}{m_e c^2}\right)^{3/2} e^{-\frac{\chi}{k_B T}} \tag{348}$$

With $\chi = 13.6\,\mathrm{eV}$ corresponding to $\sim 10^5\,\mathrm{K}$ ($1\,\mathrm{eV} \approx 10^4\,\mathrm{K}$), we would expect $x < 1$ for $T < 10^5\,\mathrm{K}$. However, there are many more photons than baryons, which leads to $x < 1$ only for $T \approx 3000\,\mathrm{K}$. This gives $z_{\mathrm{recomb}} \approx 1090$ (for $\Omega_{b,0} = 0.045, T_0 = 2.73\,\mathrm{K}$).

After this time, photons can escape or free stream, and we can observe them as the Cosmic Microwave Background. The CMB is very uniform: $\frac{\Delta T}{T} \leq 10^{-5}$ (note that CMB maps are typically logarithmic).

Why is this a problem?
Horizons are the largest causally connected regions by light rays.

The comoving horizon size is:

$$\mathrm{d}s = 0 \text{ (light)} \implies c\mathrm{d}t = a(t)\mathrm{d}\chi \implies \chi_{\mathrm{horizon}} = \int_0^t \frac{c\mathrm{d}t}{a(t)} \tag{349}$$

So we have

$$\chi_{\mathrm{horizon}}(z) = \int_0^{a=(1+z)^{-1}} \frac{c\mathrm{d}a}{a^2 H(a)}\ . \tag{350}$$

For a flat radiation dominated universe:

$$l_{\mathrm{horizon}} = a\chi_{\mathrm{horizon}} = \frac{c}{H(z)} = \frac{c}{H_0\sqrt{\Omega_{r,0}}}\frac{1}{1+z} \tag{351}$$

flat matter dominated:

$$\begin{aligned} l_{\mathrm{horizon}} =& \underbrace{a\left(\int_0^{(1+z_{eg})^{-1}} \frac{c\mathrm{d}a}{a^2 H(a)} + \int_{(1+z_{eq})^{-1}}^{(1+z)^{-1}} \frac{c\mathrm{d}a}{a^2 H(a)}\right)}_{\text{largest contribution comes from matter dominated phase}} \\ \approx& \frac{2c}{H(z)} = \frac{2c}{H_0\sqrt{\Omega_{m,0}}}\frac{a}{\sqrt{1+z}} = \frac{2c}{H_0\sqrt{\Omega_{m,0}}}\frac{1}{(1+z)^{\frac{3}{2}}} \end{aligned} \tag{352}$$

Apply this to z_{recomb}:

$$\begin{aligned} &\frac{\rho_{r,0}(1+z_{\mathrm{recomb}})^4}{\rho_{m,0}(1+z_{\mathrm{recomb}})^3} \sim 5\times 10^{-2} \Rightarrow \text{matter dominated regime} \\ \Rightarrow & l_{\mathrm{horizon}} = \frac{2c}{H_0}\Omega_{m,0}^{-\frac{1}{2}}(1+z)^{-\frac{3}{2}} \end{aligned} \tag{353}$$

Angular size of horizon:

$$\varphi_{\mathrm{horizon}} = \frac{l_{\mathrm{horizon}}}{D_A} \qquad D_A : \text{Angular diameter distance} \tag{354}$$

$$\begin{aligned}
D_A &= \frac{1}{1+z} f_\kappa(\chi_{\mathrm{em}}^{\mathrm{obs}}) = \frac{1}{1+z}\chi_{\mathrm{em}}^{\mathrm{obs}} \qquad \text{(for flat universe)} \\
&= \frac{c}{1+z}\frac{1}{H_0}\int_0^z \left[\Omega_{m,0}(1+z)^3\right]^{-\frac{1}{2}} \mathrm{d}z \\
&= \left(\frac{2c}{H_0(1+z)\Omega_m 0^{\frac{1}{2}}}\left[-\frac{1}{\sqrt{1+z}}\right]\right)_0^z \\
&= \frac{2c}{H_0 \underbrace{(1+z)}_{\approx z, z\gg 1} \Omega_{m0}^{\frac{1}{2}}} \underbrace{\left[1-\frac{1}{\sqrt{1+z}}\right]}_{\approx 1, z\gg 1} \\
&\approx \frac{2c}{H_0}\frac{1}{\Omega_{m0}^{\frac{1}{2}} z}
\end{aligned} \tag{355}$$

This gives us the angular size of the horizon at recombination:

$$\varphi_{\mathrm{horizon,recomb}} \approx \sqrt{\frac{1}{z_{\mathrm{recomb}}}} \sim 1.7^\circ \tag{356}$$

Or more generally:

$$\varphi_{\mathrm{horizon,recomb}} \approx 1.7^\circ \sqrt{\Omega_{m,0}}\,. \tag{357}$$

This is much smaller than the full sky, so how can the CMB be so uniform?

- Flatness problem:
  At high z, Λ is irrelevant in the Friedmann equations, so:

$$H^2(a) = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} = H^2(a)\left[\Omega(a) - \frac{kc^2}{a^2 H^2(a)}\right], \qquad \rho = \rho_m + \rho_r \tag{358}$$

  Thus, deviation from flatness $\Omega(a) = 1$ is:

$$|\Omega(a) - 1| = \frac{kc^2}{a^2 H^2(a)}\,. \tag{359}$$

  Since $a \propto t^{2/3}$ in matter dominated times and $a \propto t^{1/2}$ during radiation dominated times, we have:

$$|\Omega(t) - 1| \propto \begin{cases} t, & \text{radiation dominated} \\ t^{2/3}, & \text{matter dominated} \end{cases} \tag{360}$$

  Thus, any small deviation $\Omega(t_{\mathrm{early}}) \neq 1$ at early times quickly blows up! $\Omega(t_{\mathrm{early}})$ must therefore be very close to 1, which leads to a “fine-tuning problem."

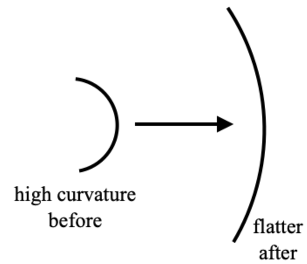
- Monopole problem:
  General unified theories predict many magnetic monopoles, but this is not observed. The number density must decrease.

- Seeds of structure formation problem:
  What seeds the perturbations that become the large structures we observe?

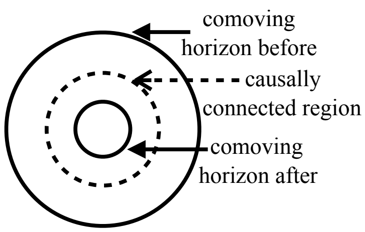
Inflation: basic ideas:

- Flatness problem:

  If $\frac{kc^2}{a^2H^2(a)}$ decreased with time for a short period, then $\Omega(a)$ would be driven towards $\Omega(a) = 1$.



- Horizon problem:
  If $\frac{kc^2}{a^2H^2(a)}$ shrinks, then $\chi \propto \frac{c}{aH(a)}$ also shrinks.



$\Rightarrow$ can explain smoothness within the observable universe.

So decreasing $\frac{1}{aH(a)}$ seems to solve two problems! The conditions for a shrinking comoving horizon:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{aH}\right) &< 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{\dot{a}}\right) &< 0\\ -\frac{c\ddot{a}}{\dot{a}^2} &< 0\\ \Rightarrow \ddot{a} &> 0\end{aligned} \tag{361}$$

We need some period of accelerated expansion. We can look at the second Friedmann equation (e.g. for acceleration):

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{3}\\ &\text{(at early times, } \Lambda = 0)\\ &= -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2})\\ &\Rightarrow p < -\frac{\rho c^2}{3} \leftarrow \text{we need sufficiently negative pressure}\\ &\Rightarrow \frac{p}{\rho c^2} < -\frac{1}{3}\end{aligned} \tag{362}$$

This also solves the monopole and seed problem! Rapid expansion would decrease the density of monopoles and blow up tiny perturbations. All problems are then solved.

Note that $\frac{\Lambda c^2}{3}$ actually corresponds to a negative pressure term. To see this more clearly, we combine both Friedmann equations to derive the energy conservation equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2 a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0 \\ \Rightarrow \dot{\rho} = -3H(a)(\rho + \frac{p}{c^2}) \end{aligned} \tag{363}$$

And for Λ with $\rho_\Lambda = \text{constant}\ (= \rho)$:

$$\rho + \frac{p}{c^2} = 0 \ \Rightarrow p = -\rho c^2 \tag{364}$$

So the equation of state parameter is

$$w = \frac{p}{\rho c^2} = -1 < -\frac{1}{3} \tag{365}$$

where $w = -1/3$ is needed for accelerated expansion as shown above. Thus, Λ leads to accelerated expansion and therefore a shrinking comoving horizon. Once Λ dominates in the Friedmann equation:

$$\begin{aligned} H^2(a) = H_0^2\Omega_\Lambda = \left(\frac{\dot{a}}{a}\right)^2 \\ \Rightarrow a \propto e^{\sqrt{\Omega_\Lambda}H_0 t} \end{aligned} \tag{366}$$

which is exponential growth.

Inflation:
Λ has all the features we want, but it:

- acts too late
- is constant, i.e. even if it acted early enough, it would not stop inflation!

How do we get all this in the early universe? We look at a homogeneous scalar field (inflation):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{367}$$

which leads to the energy-momentum tensor:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \ \Rightarrow \begin{aligned} T_{00} = \rho c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ T_{ii} = p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned} \tag{368}$$

To get $w < -1/3$, we require:

$$\begin{aligned} \frac{1}{2}\dot{\phi} - V(\phi) < -\frac{1}{3}\left(\frac{1}{2}\dot{\phi} + V(\phi)\right) \\ p < -\frac{\rho c^2}{3} \\ \Rightarrow \dot{\phi}^2 < V(\phi) \end{aligned} \tag{369}$$

i.e. the field must be moving slowly during inflation. Thus, the potential should be flat and have a minimum to stop inflation. Furthermore:

$$
\begin{aligned}
\text{Friedmann equation : } H^2 &= \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \\
\text{Energy conservation : } \dot{\rho} &= -3H(a)\left(\rho + \frac{p}{c^2}\right) \\
\dot{\rho}c^2 &= \dot{\phi}\ddot{\phi} + \dot{\phi}\frac{\mathrm{d}V}{\mathrm{d}\phi} \\
&\text{with } \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \text{ and } \frac{p}{c^2} = \frac{1}{2}\dot{\phi} - V(\phi) \\
\Rightarrow &\dot{\phi}\ddot{\phi} + \dot{\phi}\frac{dV}{d\phi} = -3H(a)\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - 3H(a)\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \\
\Rightarrow &\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = -3H(a)\dot{\phi} \\
\Rightarrow &\boxed{\ddot{\phi} + \underbrace{3H(a)}_{\text{Hubble drag}}\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi}}
\end{aligned}
\tag{370}
$$

and we get the *field evolution equation*. In a static universe, $H = 0$, and there is no Hubble drag. $\frac{\mathrm{d}V}{\mathrm{d}\phi}$ is how fast energy is extracted from inflation.

Slow roll conditions:
We approximate

$$H^2 \approx \frac{8\pi G}{3}V(\phi) \tag{371}$$

which is $\approx V_0$ and roughly constant during the slow roll, leading to exponential growth. We also have

$$3H\dot{\phi} \approx -\frac{\mathrm{d}V}{\mathrm{d}\phi} \text{ (with } \ddot{\phi} \approx 0) \tag{372}$$

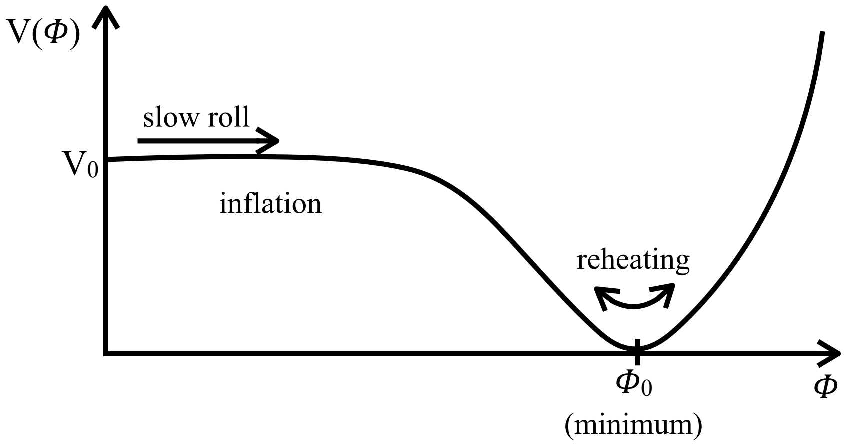
which is equivalent to:

$$
\begin{gathered}
\dot{\phi}^2 \ll V \\
\text{and} \\
\frac{\mathrm{d}}{\mathrm{d}t}\dot{\phi}^2 \ll \frac{\mathrm{d}V}{\mathrm{d}t} \Rightarrow \ddot{\phi} \ll \frac{\mathrm{d}V}{\mathrm{d}\phi}
\end{gathered}
\tag{373}
$$

This can be rewritten in slow roll parameters:

$$
\begin{aligned}
\epsilon &:= \frac{1}{24\pi G}\left(\frac{V'}{V}\right)^2 \ll 1 \\
\eta &:= \frac{1}{8\pi G}\left(\frac{V''}{V}\right) \ll 1
\end{aligned}
\tag{374}
$$

As long as these conditions are valid, inflation will go on. The slow roll potential is:



During reheating, the inflation field decays through coupling to ordinary matter ("reheat universe").

Since

$$H^2 = \frac{8\pi G}{3} V(\phi) \approx \frac{8\pi G}{3} V_0 \tag{375}$$

during inflation, large values of ϕ_0 and V_0 lead to more inflation (longer slow roll).

1.E Basic story of cosmology

Main ingredients:

- metric (geometry)
- Friedmann equations (dynamics)
- distances (connection to observations)
- horizons (evidence for inflation)

Emerging story

a) $t = 0$: Big Bang

b) $t \sim 10^{-34}$ s: inflation

c) T decreases as $T \propto (1 + z)$

d) $z \approx 3200$: transition from radiation to matter domination

e) $z \approx 1100$: recombination

f) Structure formation is nonlinear. First stars and galaxies...

g) $z \approx 0.33$: transition from matter to Λ domination

The first five stages here are optically thick to photons, while later is optically thin and potentially observable.

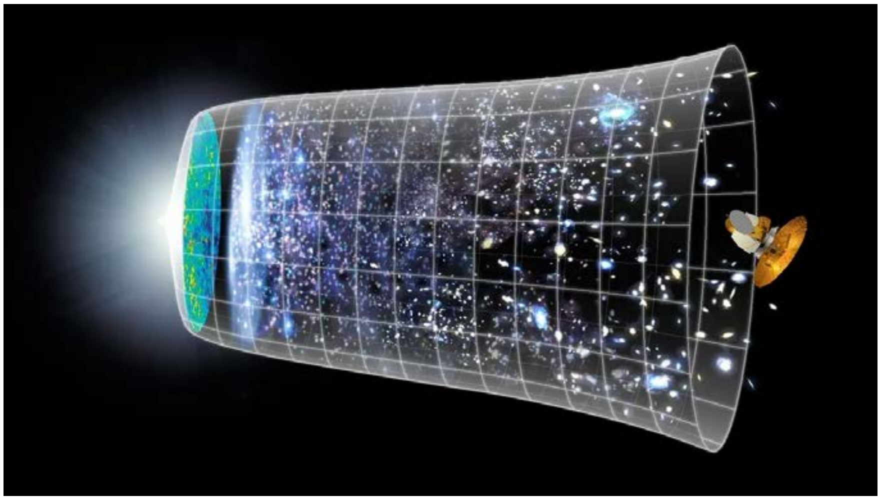


Image: NASA / WMAP Science Team. Image is in the public domain.

2 Structure formation

So far, we have assumed a uniform cosmology. We now add perturbations to study the growth of structure.

2.A Linear perturbation theory

There are small perturbations at early times. The Universe consists of matter (dark matter and baryons) and radiation. Λ and curvature are unimportant early on.

Basic equations:

- non-relativistic matter (dark matter, baryons) is important in the matter-dominated regime:

$$\begin{aligned} \text{continuity equation}: &\ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \text{momentum equation}: &\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho} + \vec{\nabla} \phi \\ \text{Poisson's equation}: &\ \vec{\nabla}^2 \phi = 4\pi G \rho \end{aligned} \tag{376}$$

- relativistic matter (radiation)

$$\begin{aligned} \text{continuity equation}: &\ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \Big((\rho + \frac{p}{c^2}) \vec{v} \Big) = 0 \\ \text{momentum equation}: &\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p}{\rho + \frac{p}{c^2}} + \vec{\nabla} \phi \\ \text{Poisson's equation}: &\ \vec{\nabla}^2 \phi = 4\pi G (\rho + \frac{3p}{c^2}) \end{aligned} \tag{377}$$

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