The first five stages here are optically thick to photons, while later is optically thin and potentially observable.

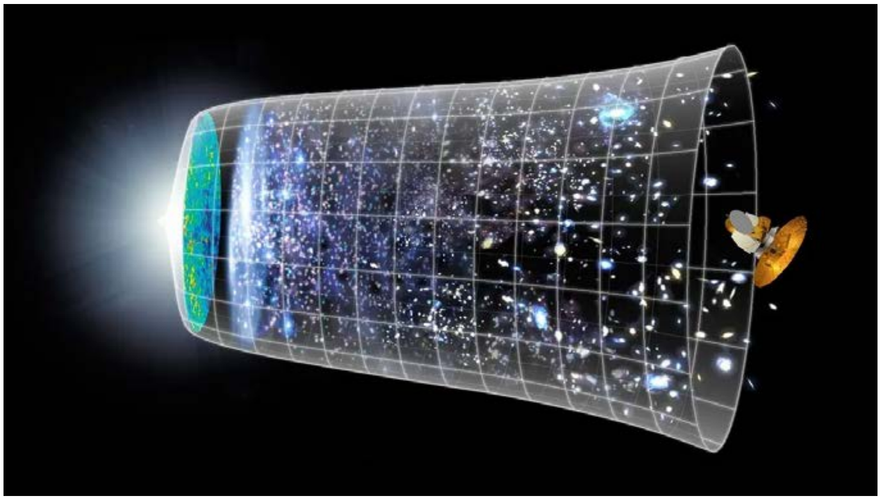

Image: NASA / WMAP Science Team. Image is in the public domain.

2 Structure formation

So far, we have assumed a uniform cosmology. We now add perturbations to study the growth of structure.

2.A Linear perturbation theory

There are small perturbations at early times. The Universe consists of matter (dark matter and baryons) and radiation. Λ and curvature are unimportant early on.

Basic equations:

- non-relativistic matter (dark matter, baryons) is important in the matter-dominated regime:

$$\begin{aligned} \text{continuity equation}: & \ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \text{momentum equation}: & \ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \vec{\nabla}\phi \\ \text{Poisson's equation:} & \ \vec{\nabla}^2 \phi = 4\pi G \rho \end{aligned} \tag{376}$$

- relativistic matter (radiation)

$$\begin{aligned} \text{continuity equation}: & \ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left((\rho + \frac{p}{c^2})\vec{v} \right) = 0 \\ \text{momentum equation}: & \ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho + \frac{p}{c^2}} + \vec{\nabla}\phi \\ \text{Poisson's equation:} & \ \vec{\nabla}^2 \phi = 4\pi G (\rho + \frac{3p}{c^2}) \end{aligned} \tag{377}$$

Notes:

- non-relativistic
  - Dark matter follows the collisionless Boltzmann equation; $1^{\text{st}}/2^{\text{nd}}$ equations only hold for moments (Jeans equation).
  - For dark matter, there is no well-defined velocity field $\vec{v}(\vec{x}, t)$ due to multistream. $\vec{v}(\vec{x}, t)$ is just an average.
  - Nevertheless, it recovers the correct growth rate for large scales $> \lambda_J$ when pressure can be neglected.
- relativistic
  - gravitational source terms include pressure terms.
  - For pure radiation: $p = \frac{\rho c^2}{3}$.

This leads to the perturbation equation where some small perturbation δ evolves in a smooth background density $\bar{\rho}$:

$$\delta = \frac{\Delta\rho}{\bar{\rho}} = \frac{\rho - \bar{\rho}}{\bar{\rho}} \ . \tag{378}$$

Perturbation equations:

$$\begin{aligned} \text{non-relativistic: } \ddot{\delta} + 2H\dot{\delta} &= \left(4\pi G\bar{\rho}\delta + \frac{v_s^2 \vec{\nabla}^2 \delta}{a^2}\right) \\ \text{relativistic: } \ddot{\delta} + 2H\dot{\delta} &= \left(\frac{32}{3}\pi G\bar{\rho}\delta + \frac{v_s^2 \vec{\nabla}^2 \delta}{a^2}\right) \end{aligned} \tag{379}$$

where

$$\begin{aligned} \delta &= \delta(\vec{x}, t) \\ \vec{x} &: \text{comoving coordinates} \\ \vec{r} &: a\vec{x} \text{ physical coordinates} \\ \vec{\nabla} &= \frac{\partial}{\partial \vec{x}} \\ v_s &= \begin{cases} c_s, & \text{non-relativistic baryons} \\ \sigma, & \text{non-relativistic dark matter} \\ \dfrac{c}{\sqrt{3}}, & \text{relativistic radiation} \end{cases} \end{aligned} \tag{380}$$

Fourier representation:

For comoving coordinate $\vec{x}$ and comoving wave number $\vec{k}$:

$$\begin{aligned}\delta(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3}\hat{\delta}(\vec{k},\delta)e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k},t) &= \int d^3x\delta(\vec{x},t)e^{+i\vec{k}\cdot\vec{x}}\end{aligned} \tag{381}$$

and we get

$$\begin{aligned}\text{non-relativistic: } \ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= \hat{\delta}\left(4\pi G\bar{\rho} - \frac{v_s^2k^2}{a^2}\right) \\ \text{relativistic: } \ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= \hat{\delta}\left(\frac{32}{3}\pi G\bar{\rho} - \frac{v_s^2k^2}{a^2}\right) .\end{aligned} \tag{382}$$

Growing modes:
For a static background, $H = 0$ and:

$$\ddot{\hat{\delta}} + w_0^2\hat{\delta} = 0 \tag{383}$$

with

$$w_0^2 = \begin{cases} \dfrac{v_s^2k^2}{a^2} - 4\pi G\bar{\rho}, & \text{non-relativistic} \\ \dfrac{v_s^2k^2}{a^2} - \dfrac{32}{3}\pi G\bar{\rho}, & \text{relativistic}\end{cases} \tag{384}$$

For physical wave number $\tilde{k} = k/a$, we get oscillation for:

$$\tilde{k} \geq \tilde{k}_J = \begin{cases} \dfrac{2\sqrt{\pi G\bar{\rho}}}{v_s}, & \text{non-relativistic} \\ \dfrac{\sqrt{\frac{32}{3}}\sqrt{\pi G\bar{\rho}}}{v_s}, & \text{relativistic}\end{cases} \tag{385}$$

and growth (no oscillations) for modes with lengths $l = \frac{2\pi}{\tilde{k}}$ greater than λ_J:

$$l \geq \lambda_J = \frac{2\pi}{\tilde{k}_J} \propto \frac{v_s}{\sqrt{\pi G\bar{\rho}}} \tag{386}$$

where $\tilde{k} = k/a$ is in physical units. We can make this more general for $H \neq 0$ and neglecting the pressure terms for $l \geq \lambda_J$ and we get:

$$\begin{aligned}\ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= 4\pi G\bar{\rho}\hat{\delta}, && \text{non-relativistic} \\ \ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= \frac{32}{3}\pi G\bar{\rho}\hat{\delta}, && \text{relativistic}\end{aligned} \tag{387}$$

Now for $\Omega = 1$, we have the critical density as the background density for the radiation and matter dominated regime:

$$\bar{\rho} = \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \tag{388}$$

such that

$$\begin{aligned}\ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= \frac{3}{2}H^2\hat{\delta}, \text{ matter dominated } \Omega = 1 \\ \ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} &= 4H^2\hat{\delta}, \text{ radiation dominated } \Omega = 1\end{aligned} \tag{389}$$

Now:

$$H = \frac{\dot{a}}{a} = \begin{cases} \frac{2}{3t}, & \text{matter dominated} \\ \frac{1}{2t}, & \text{radiation dominated} \end{cases} \tag{390}$$

We now assume $\hat{\delta}(\vec{k}, t) \propto t^n$. Then:

$$\begin{aligned} n^2 + \frac{n}{3} - \frac{2}{3} = 0, & \qquad \text{matter dominated} \\ n^2 - 1 = 0, & \qquad \text{radiation dominated} \end{aligned} \tag{391}$$

$$\Rightarrow \begin{aligned} n = -1, \frac{2}{3}, & \qquad \text{matter dominated} \\ n = -1, +1, & \quad \text{radiation dominated} \end{aligned} \tag{392}$$

that correspond to negative decaying modes, which are unimportant since the perturbations vanish, and positive growing modes. This gives us:

$$\boxed{\hat{\delta} \propto \begin{cases} a, & \text{matter dominated} \\ a^2, & \text{radiation dominated} \end{cases}} \tag{393}$$

In general, we write $\delta = D\delta_0$ or $\hat{\delta} = D\hat{\delta}_0$ where D is the growth factor such that $D(z = 0) = 1$.

Growth of baryons vs. cold dark matter:
Growth of perturbations occurs for $\lambda > \lambda_J$ or $M > M_J = \frac{4}{3}\pi\bar{\rho}\lambda_J^3$.
Baryons:

- Until recombination, there is strong coupling between photons and electrons.
- Before recombination

$$\begin{aligned} c_s^2 &= \frac{\partial p}{\partial \rho}, \; p = \frac{1}{3}\rho c^2, \; \text{(radiation)} \\ \Rightarrow c_s &= \frac{c}{\sqrt{3}}\,. \end{aligned} \tag{394}$$

- After recombination

$$\begin{aligned} c_s^2 &= \frac{\partial p}{\partial \rho}, \; p = \frac{\rho}{mkT}, \; T = 2.71\,\mathrm{K}(1 + \mathrm{z}), \; \text{(ideal gas)} \\ \Rightarrow c_s &= \sqrt{\frac{kT}{m}} \approx 5\,\mathrm{km/s} \end{aligned} \tag{395}$$

- So
$$z \gg 1100: \;\; M_J \geq 10^{16}\,\mathrm{M}_\odot \\ z < 1100: \;\; M_J \lesssim 10^{5}\,\mathrm{M}_\odot \tag{396}$$
since after recombination, the photon pressure support is removed.

- Structure can only form after $z \sim 1000$.

- if structure can only grow from $z \sim 1000$, δ will be amplified by $\sim 10^3$ (since matter dominated growth is $\propto a$). BUT: the CMB has $\delta \sim 10^{-5}$ and $10^{-5} \times 10^3 \sim 10^{-2}$ today, which is much less than what we observe in the low redshift universe. This theory of structure growth is not sufficient.

- Solution: dark matter must have clumped before and baryons fall into dark matter wells.

Cold dark matter:

- CDM is very cold, so it has a tiny velocity dispersion σ. This means that M_J is tiny and collapse on all scales is possible.

- CDM does not interact with radiation, so it can grow before recombination.

Cold dark matter is needed to make structure formation work!

2.B Growth of linear perturbations

Full general relativity treatment can be used to study growth beyond the horizon scale. Modes outside the horizon can always grow (no causal contact):
$$\delta \propto \begin{cases} a \propto t^{2/3}, & \text{matter dominated} \\ a^2 \propto t, & \text{radiation dominated} \end{cases} \tag{397}$$
Once a mode enters the horizon, its growth changes (note that we have perturbations on different length scales).
Baryons:
Baryons have a finite Jeans length before recombination:
$$\lambda_J = \frac{c}{\sqrt{G\bar{\rho}}}\sqrt{\frac{\pi}{3}}, \;\; (c_s = \frac{c}{\sqrt{3}}) \tag{398}$$
so modes with $l < \lambda_J$ have no growth. However, this is only if l is also within the horizon.

The growth of the physical horizon is (for $\Omega = 1$):
$$\begin{aligned} l_{\text{horizon}} = a\int_0^t \frac{c\mathrm{d}t}{a(t)} &= \begin{cases} 2ct, & \text{radiation dominated; } a \propto t^{1/2} \\ 3ct, & \text{matter dominated; } a \propto t^{2/3} \end{cases} \\ &= \begin{cases} \dfrac{c}{\sqrt{G\bar{\rho}}}\sqrt{\dfrac{3}{8\pi}}, & \text{radiation dominated} \\ \dfrac{c}{\sqrt{G\bar{\rho}}}\sqrt{\dfrac{3}{2\pi}}, & \text{matter dominated} \end{cases} \end{aligned} \tag{399}$$