- So

$$\begin{aligned} z \gg 1100: \ M_J &\geq 10^{16}\,\mathrm{M}_\odot \\ z < 1100: \ M_J &\lesssim 10^{5}\,\mathrm{M}_\odot \end{aligned} \tag{396}$$

  since after recombination, the photon pressure support is removed.

- Structure can only form after $z \sim 1000$.
- if structure can only grow from $z \sim 1000$, δ will be amplified by $\sim 10^3$ (since matter dominated growth is $\propto a$). BUT: the CMB has $\delta \sim 10^{-5}$ and $10^{-5} \times 10^3 \sim 10^{-2}$ today, which is much less than what we observe in the low redshift universe. This theory of structure growth is not sufficient.
- Solution: dark matter must have clumped before and baryons fall into dark matter wells.

Cold dark matter:

- CDM is very cold, so it has a tiny velocity dispersion σ. This means that M_J is tiny and collapse on all scales is possible.
- CDM does not interact with radiation, so it can grow before recombination.

Cold dark matter is needed to make structure formation work!

2.B Growth of linear perturbations

Full general relativity treatment can be used to study growth beyond the horizon scale. Modes outside the horizon can always grow (no causal contact):

$$\delta \propto \begin{cases} a \propto t^{2/3}, & \text{matter dominated} \\ a^2 \propto t, & \text{radiation dominated} \end{cases} \tag{397}$$

Once a mode enters the horizon, its growth changes (note that we have perturbations on different length scales).
Baryons:
Baryons have a finite Jeans length before recombination:

$$\lambda_J = \frac{c}{\sqrt{G\bar{\rho}}}\sqrt{\frac{\pi}{3}}, \ (c_s = \frac{c}{\sqrt{3}}) \tag{398}$$

so modes with $l < \lambda_J$ have no growth. However, this is only if l is also within the horizon.

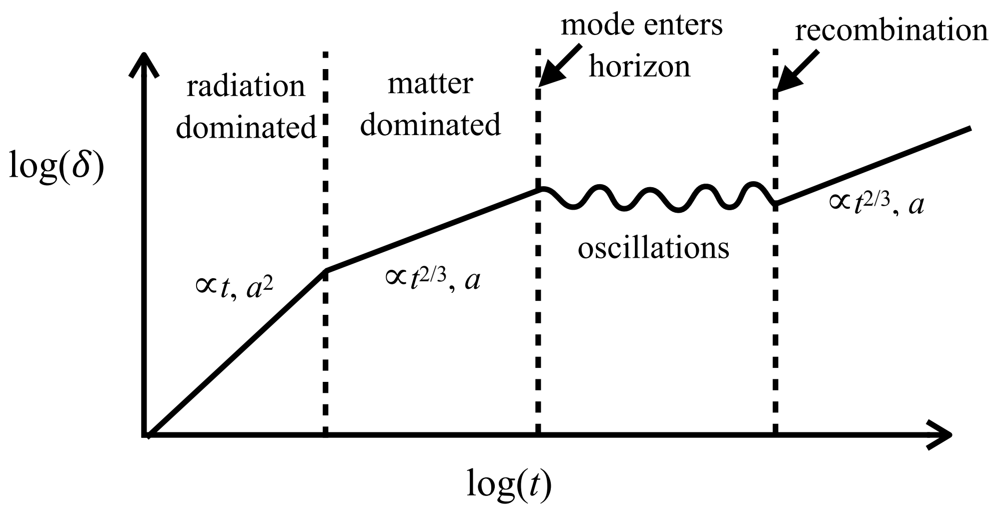
The growth of the physical horizon is (for $\Omega = 1$):

$$\begin{aligned} l_{\text{horizon}} = a\int_0^t \frac{c\mathrm{d}t}{a(t)} &= \begin{cases} 2ct, & \text{radiation dominated; } a \propto t^{1/2} \\ 3ct, & \text{matter dominated; } a \propto t^{2/3} \end{cases} \\ &= \begin{cases} \dfrac{c}{\sqrt{G\bar{\rho}}}\sqrt{\dfrac{3}{8\pi}}, & \text{radiation dominated} \\ \dfrac{c}{\sqrt{G\bar{\rho}}}\sqrt{\dfrac{3}{2\pi}}, & \text{matter dominated} \end{cases} \end{aligned} \tag{399}$$

where we used

$$H^2 = \frac{8\pi G\rho}{3} \text{ (for } \rho = \rho_{\rm crit} = \frac{3H^2}{8\pi G}) = \begin{cases} \frac{1}{4t^2}, & \text{radiation} \\ \frac{4}{at^2}, & \text{matter} \end{cases} \tag{400}$$

$l_{\rm horizon} < \lambda_J$: as soon as mode length l enters the horizon, it will oscillate! So before recombination, perturbations can grow if $l > l_{\rm horizon}$, otherwise they will oscillate.



The horizon and Jeans mass grow as we have seen before: $M_J \sim 10^{16}{\rm M}_\odot$ at $z \sim 1000$, so all modes smaller than $10^{16}{\rm M}_\odot$ entered the horizon before recombination and therefore start to oscillate and stop growing.

There is also another problem for those modes: *Silk damping*! Before decoupling, photons do not free stream because of Thomson scattering off free electrons. The mean free path gets large towards recombination. So:

- $M < M_J \sim 10^{16}{\rm M}_\odot$ perturbations oscillate due to photon pressure.
- Photons can diffuse out of potential wells and take baryons with them (electrons through Thomson scattering and protons through Coulomb interactions), which erases perturbations.

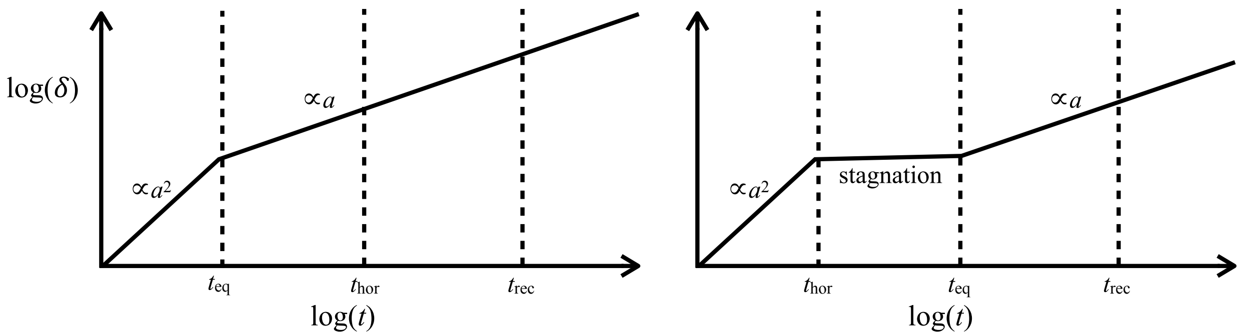
The net effect is that all perturbations $\sim 10^{12}{\rm M}_\odot$ (Silk mass) are damped and erased!

Cold dark matter:
Cold dark matter has essentially zero Jeans mass, so all modes can already grow. However, for subhorizon modes in the radiation dominated epoch, δ ∼constant (stagnation). Because

the expansion rate is higher than the growth rate, we get:

$$\begin{aligned} \text{expansion timescale: } \tau_{\text{Hubble}} &\approx \frac{1}{\sqrt{G\rho_r}} \\ \text{collapse timescale: } \tau_{\text{Jeans}} &\approx \frac{1}{\sqrt{G\rho_m}} \end{aligned} \tag{401}$$

and $\tau_{\text{Hubble}} \ll \tau_{\text{Jeans}}$ if $\rho_r \gg \rho_m$. So modes entering the horizon during the radiation dominated phase are frozen (but not damped through something like Silk damping). After recombination, baryons can fall into CDM wells and grow.



A mode that enters the horizon at t_{hor} after matter-radiation equality at t_{eq} will always grow. Modes that enter the horizon during the radiation dominated regime will stagnate until matter domination.

Cold dark matter is then the main driver of structure formation since it there is time for CDM perturbations to grow large enough. Without CDM, structure formation is not possible.

2.C Statistical measures of structure

We see structure on different scales. We can use the power spectrum $P(k)$ to describe this. Reminder:

$$\begin{aligned} \delta(\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \hat{\delta}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \\ \hat{\delta}(\vec{k}) &= \int d^3x \delta(\vec{x}) e^{+i\vec{k}\cdot\vec{x}} \end{aligned} \tag{402}$$

Variance and the power spectrum:
average:

$$\langle \delta \rangle = \int d^3x \delta(\vec{x}) = 0 \tag{403}$$

variance:

$$\begin{aligned} \sigma^2 = \left\langle \delta^2 \right\rangle - \left\langle \delta \right\rangle^2 &= \left\langle \delta^2 \right\rangle > 0 \\ \left\langle \delta^2 \right\rangle = \int \mathrm{d}^3x \delta^2(\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} |\hat{\delta}(\vec{k})|^2 \end{aligned} \tag{404}$$

If we assume homogeneity and isotropy, $\vec{k} \to k = |\vec{k}|$ and $\mathrm{d}^3\vec{k} = 4\pi k^2 \mathrm{d}k$. Then we get:

$$\begin{aligned} \sigma^2 &= \frac{1}{2\pi^2}\int |\hat{\delta}(k)|^2 k^2 \mathrm{d}k \\ &=: \frac{1}{2\pi^2}\int P(k) k^2 \mathrm{d}k \\ &\text{with } P(k) = |\hat{\delta}(k)|^2 \end{aligned} \tag{405}$$

Notes:

- $P(k)$ and σ are functions of time since $\hat{\delta}(k)$ grows ($\sigma = D\sigma_0$).
- The initial power spectrum is the primordial power spectrum set at the end of inflation. The general form is
$$P(k) = Ak^n \tag{406}$$
which is a power law and is scale-free. According to predictions from inflation, $n \approx 1$.

Measuring $P(k)$ and galaxy clustering:
If we assume galaxies trace the mass perturbations, what is the probability dP that we find two galaxies in volumes dV_1 and dV_2 at a distance r from each other?

$$\begin{aligned} \mathrm{d}P &= n_0(1+\delta(\vec{x}))\mathrm{d}V_1 \cdot n_0(1+\delta(\vec{x}+\vec{r}))\mathrm{d}V_2 \\ &= n_0^2(1 + \underbrace{\delta(\vec{x})}_{=0} + \underbrace{\delta(\vec{x}+\vec{r})}_{=0} + \delta(\vec{x})\delta(\vec{x}+\vec{r}))\mathrm{d}V_1\mathrm{d}V_2 \\ &= n_0^2(1+\xi(r))\mathrm{d}V_1\mathrm{d}V_2 \end{aligned} \tag{407}$$

where $\delta(\vec{x})$ and $\delta(\vec{x}+\vec{r})$ are zero on average and $\vec{r} \to r$ due to isotropy. ξ is the *two-point correlation function* and is related to $P(k)$:

$$\begin{aligned} \xi(r) &= \int \mathrm{d}^3\vec{x}\,\delta(\vec{x})\delta(\vec{x}+\vec{r}) \\ &= \int \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\hat{\delta}(\vec{k})e^{-i\vec{k}\cdot\vec{x}} \underbrace{\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\hat{\delta}(\vec{k}')e^{-i\vec{k}'\cdot(\vec{x}+\vec{r})}}_{\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\hat{\delta}(\vec{k}')e^{+i\vec{k}'\cdot(\vec{x}+\vec{r})}\ (\delta\ \text{real})} \\ &= \left(\frac{1}{(2\pi)^3}\right)^2 \int \mathrm{d}^3\vec{x} \int\int \mathrm{d}^3k\, d^3k'\ \hat{\delta}(\vec{k})\hat{\delta}(\vec{k}')e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}e^{-i\vec{k}\vec{r}} \\ &\qquad \text{using } \frac{1}{(2\pi)^3}\int \mathrm{d}^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta(\vec{k}-\vec{k}') \\ &= \frac{1}{(2\pi)^3}\int |\hat{\delta}(\vec{k})|^2 e^{i\vec{k}\cdot\vec{r}}\mathrm{d}^3k \\ &= \frac{1}{(2\pi)^3}\int |\hat{\delta}(k)|^2 e^{i\vec{k}\cdot\vec{r}}\mathrm{d}^3k \quad \text{where } \vec{k} \to k \text{ from isotropy} \\ &= \frac{1}{(2\pi)^3}\int P(k) e^{i\vec{k}\cdot\vec{r}}\mathrm{d}^3k \\ \Rightarrow & \boxed{\xi(r) = \frac{1}{(2\pi)^3}\int P(k) e^{i\vec{k}\cdot\vec{r}}\mathrm{d}^3k}\,. \end{aligned} \tag{408}$$

Observationally:

$$\xi(r) \approx \left(\frac{r}{r_0}\right)^{-1.8} \tag{409}$$

with $r_0 \approx 5\,h^{-1}$ Mpc for galaxies. Different objects have a different r_0, and more massive objects are more clustered, e.g. the cluster-cluster correlation function differs from the galaxy-galaxy correlation function: $\xi_{cc} \approx 20\xi_{gg}$.

2.D Form of the primordial power spectrum

There is no scale in the power spectrum $P(k) = Ak^n$. We want to know what n and A are. Initially, fluctuations on different scales should have the same amplitude on different scales.

Power spectrum index:
Fluctuations on certain mass or length scales are (0 is large scale, $k_{\max}$ is the smallest scale):

$$\begin{aligned}\sigma^2 &\approx \int_0^{k_{\max}} P(k)k^2 \mathrm{d}k \\ &= \int_0^{k_{\max}} Ak^{n+2}\mathrm{d}k \propto k_{\max}^{n+3} \\ \Rightarrow \sigma &\propto k_{\max}^{\frac{1}{2}(n+3)} \text{ or } \sigma \propto k^{\frac{1}{2}(n+3)}\end{aligned} \tag{410}$$

For mass, we get:

$$\begin{aligned}M \propto R^3 \propto k^{-3} &\Rightarrow k \propto M^{-1/3} \\ &\Rightarrow \sigma \propto M^{-\frac{1}{6}(n+3)}\end{aligned} \tag{411}$$

So:

$$\sigma \propto \begin{cases} k^{\frac{1}{2}(n+3)} \\ M^{-\frac{1}{6}(n+3)} \end{cases} \tag{412}$$

Does this tell us something about n? Modes can always grow outside the horizon, but we do not want "special" modes. All modes should therefore have the same σ, i.e. the same strength/fluctuation amplitude, when they enter the horizon.

The horizon mass, i.e. the mass within the horizon, is:

$$\begin{aligned}M_h &\propto \rho_m r_h^3 \\ \rho_m &\propto (1+z)^3\end{aligned} \tag{413}$$

and

$$\begin{aligned}r_h &\propto \begin{cases} a^2 = (1+z)^{-2}, & \text{radiation dominated} \\ a^{3/2} = (1+z)^{-3/2}, & \text{matter dominated} \end{cases} \\ \Rightarrow M_h &\propto \begin{cases} (1+z_h)^{-3}, & \text{radiation dominated} \\ (1+z_h)^{-3/2}, & \text{matter dominated} \end{cases}\end{aligned} \tag{414}$$