Observationally:

$$\xi(r) \approx \left(\frac{r}{r_0}\right)^{-1.8} \tag{409}$$

with $r_0 \approx 5\,h^{-1}$ Mpc for galaxies. Different objects have a different r_0, and more massive objects are more clustered, e.g. the cluster-cluster correlation function differs from the galaxy-galaxy correlation function: $\xi_{cc} \approx 20\xi_{gg}$.

2.D Form of the primordial power spectrum

There is no scale in the power spectrum $P(k) = Ak^n$. We want to know what n and A are. Initially, fluctuations on different scales should have the same amplitude on different scales.

Power spectrum index:
Fluctuations on certain mass or length scales are (0 is large scale, $k_{\max}$ is the smallest scale):

$$\begin{aligned} \sigma^2 &\approx \int_0^{k_{\max}} P(k)k^2 \mathrm{d}k \\ &= \int_0^{k_{\max}} Ak^{n+2}\mathrm{d}k \propto k_{\max}^{n+3} \\ \Rightarrow \sigma &\propto k_{\max}^{\frac{1}{2}(n+3)} \text{ or } \sigma \propto k^{\frac{1}{2}(n+3)} \end{aligned} \tag{410}$$

For mass, we get:

$$\begin{aligned} M \propto R^3 \propto k^{-3} &\Rightarrow k \propto M^{-1/3} \\ &\Rightarrow \sigma \propto M^{-\frac{1}{6}(n+3)} \end{aligned} \tag{411}$$

So:

$$\sigma \propto \begin{cases} k^{\frac{1}{2}(n+3)} \\ M^{-\frac{1}{6}(n+3)} \end{cases} \tag{412}$$

Does this tell us something about n? Modes can always grow outside the horizon, but we do not want "special" modes. All modes should therefore have the same σ, i.e. the same strength/fluctuation amplitude, when they enter the horizon.

The horizon mass, i.e. the mass within the horizon, is:

$$\begin{aligned} M_h &\propto \rho_m r_h^3 \\ \rho_m &\propto (1+z)^3 \end{aligned} \tag{413}$$

and

$$\begin{aligned} r_h &\propto \begin{cases} a^2 = (1+z)^{-2}, & \text{radiation dominated} \\ a^{3/2} = (1+z)^{-3/2}, & \text{matter dominated} \end{cases} \\ \Rightarrow M_h &\propto \begin{cases} (1+z_h)^{-3}, & \text{radiation dominated} \\ (1+z_h)^{-3/2}, & \text{matter dominated} \end{cases} \end{aligned} \tag{414}$$

σ grows:

$$\sigma \propto \delta$$
$$\sigma \propto \begin{cases} a^2 = (1+z)^{-2}, \text{ radiation dominated} \\ a^{3/2} = (1+z)^{-1}, \text{ matter dominated} \end{cases} \tag{415}$$

We now find σ of the horizon mass, i.e. the fluctuation strength once this mode enters.

- radiation dominated case:

$$\sigma(z_h) = \sigma(z_p)\left(\frac{1+z_p}{1+z_h}\right)^2 \propto \sigma(z_p)(1+z_h)^{-2} \tag{416}$$

  where z_h is the redshift once mass M is within the horizon, and z_p is the redshift at the end of inflation. Then

$$\begin{aligned} \sigma(z_p) &\propto M^{-\frac{1}{6}(n+3)} \\ M_h = M &\propto (1+z_h)^{-3} \Rightarrow (1+z_h)^{-2} \propto M^{2/3} \end{aligned} \tag{417}$$

  so we find:

$$\sigma(z_h) \propto M^{-\frac{1}{6}(n+3)} M^{2/3} = M^{-(\frac{1}{2}+\frac{n-4}{6})} \tag{418}$$

- matter dominated case
  This follows the same calculation, so we get the same result and the fluctuation of a mode once it enters the horizon is:

$$\sigma(z_h) \propto M^{-(\frac{1}{2}+\frac{n-4}{6})} \tag{419}$$

Now, we do not want "special" modes, so $\sigma(z_h)$ should not depend on n! We get $n \approx 1$ according to the Harrison-Zel'dovich spectrum.

Power spectrum amplitude:
n can be calculated with theory from inflation, but the amplitude comes from observations. We measure the number of fluctuations in galaxy surveys within a sphere of $8\,\mathrm{Mpc}/h$, or σ_8. The fluctuations in galaxies are not exactly the fluctuations in mass:

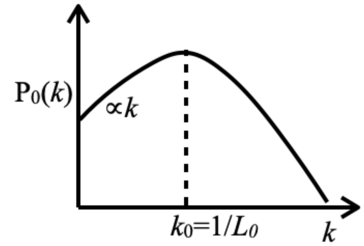
$$\sigma_{8,\mathrm{gal}} = b\sigma_{8,\mathrm{mass}} \tag{420}$$

where b is the *bias* of the galaxy clustering compared to the mass fluctuations. Observationally, $\sigma_{8,\mathrm{gal}} \approx 1$. From WMAP and SDSS, we have:

$$\begin{aligned} n &= 0.953 \pm 0.016 \\ \sigma_8 &= 0.756 \pm 0.035 \end{aligned} \tag{421}$$

Transfer function:
We found that modes entering the horizon during the radiation dominated phase do not grow

(stagnation). The primordial power spectrum is therefore modified by the transfer function:



$$
\begin{aligned}
&P_0(k) = (Ak)T^2(k),\\
&T(K) \approx \begin{cases} 1, \dfrac{1}{k} \gg L_0 \\ \dfrac{1}{k^2}, \dfrac{1}{k} \ll L_0 \end{cases}
\end{aligned}
\tag{422}
$$

where L_0 is the comoving horizon at z_{equality}.

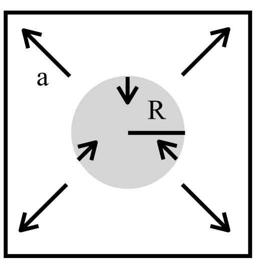
2.E Nonlinear evolution: spherical collapse

For $\delta \ll 1$, we can use linear perturbation theory, but for $\delta \sim 1$, nonlinear evolution begins and halos form. This requires simulations.

Halos:

- A distribution of dark matter as a collection of nearly spherical overdense clouds to form halos.
- We study the dynamics of spherical, homogeneous overdensities for a basic understanding. This is the spherical collapse model.

Spherical collapse model:
We consider an overdense sphere in an Einstein-de Sitter cosmology. The overdensity will eventually reach a maximum radius and then collapse to a virialized halo because the gravity within the overdensity is stronger.



$$
\begin{aligned}
H &= H_0 a^{-3/2} && \text{Friedmann equation for Einsten-de Sitter}\\
x &= \frac{a}{a_{\text{ta}}} && a_{\text{ta}} \text{ is the scale factor at maximum expansion}\\
y &= \frac{R}{R_{\text{ta}}} && \text{radius in units of maximum radius}
\end{aligned}
\tag{423}
$$

We can simplify:

$$
\begin{gathered}
\tau = H_{\text{ta}} t \quad (\text{with } H_{\text{ta}} = H_0 a_{\text{ta}}^{-3/2})\\
\Rightarrow x' = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1}{H_{\text{ta}}}\frac{\dot{a}}{a_{\text{ta}}} = \frac{H}{H_{\text{ta}}} x = x^{-1/2}\\
\left(\text{using } \frac{H}{H_{\text{ta}}} = \frac{H_0 a^{-3/2}}{H_0 a_{\text{ta}}^{-3/2}} = \frac{a^{-3/2}}{a_{\text{ta}}^{-3/2}} = x^{-3/2} \text{ for the final equality}\right)
\end{gathered}
\tag{424}
$$

So

$$
\boxed{x' = x^{-1/2}}
\tag{425}
$$

8.902 Astrophysics II
Fall 2023