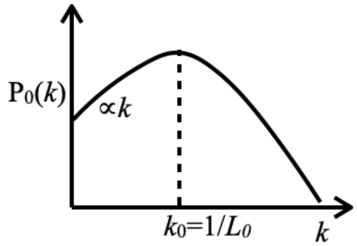
(stagnation). The primordial power spectrum is therefore modified by the transfer function:




$$
\begin{aligned}
&P_0(k) = (Ak)T^2(k), \\
&T(K) \approx \begin{cases} 1, \dfrac{1}{k} \gg L_0 \\ \dfrac{1}{k^2}, \dfrac{1}{k} \ll L_0 \end{cases}
\end{aligned}
\tag{422}
$$

where $L_0$ is the comoving horizon at $z_{\text{equality}}$.

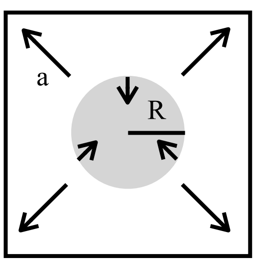
## 2.E Nonlinear evolution: spherical collapse

For $\delta \ll 1$, we can use linear perturbation theory, but for $\delta \sim 1$, nonlinear evolution begins and halos form. This requires simulations.

Halos:

- A distribution of dark matter as a collection of nearly spherical overdense clouds to form halos.
- We study the dynamics of spherical, homogeneous overdensities for a basic understanding. This is the spherical collapse model.

**Spherical collapse model**:
We consider an overdense sphere in an Einstein-de Sitter cosmology. The overdensity will eventually reach a maximum radius and then collapse to a virialized halo because the gravity within the overdensity is stronger.




$$
\begin{aligned}
H &= H_0 a^{-3/2} && \text{Friedmann equation for Einsten-de Sitter} \\
x &= \frac{a}{a_{\text{ta}}} && a_{\text{ta}} \text{ is the scale factor at maximum expansion} \\
y &= \frac{R}{R_{\text{ta}}} && \text{radius in units of maximum radius}
\end{aligned}
\tag{423}
$$

We can simplify:

$$
\begin{aligned}
&\tau = H_{\text{ta}} t \quad (\text{with } H_{\text{ta}} = H_0 a_{\text{ta}}^{-3/2}) \\
\Rightarrow\ &x' = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1}{H_{\text{ta}}}\frac{\dot{a}}{a_{\text{ta}}} = \frac{H}{H_{\text{ta}}} x = x^{-1/2} \\
&(\text{using } \frac{H}{H_{\text{ta}}} = \frac{H_0 a^{-3/2}}{H_0 a_{\text{ta}}^{-3/2}} = \frac{a^{-3/2}}{a_{\text{ta}}^{-3/2}} = x^{-3/2} \text{ for the final equality})
\end{aligned}
\tag{424}
$$

So

$$
\boxed{x' = x^{-1/2}}
\tag{425}
$$

We use the Newtonian equation of motion for the radius $R$:

$$\ddot{R} = -\frac{GM}{R^2} = -\underbrace{\frac{4\pi}{3}\rho_{\mathrm{ta}} R_{\mathrm{ta}}^3}_{\text{enclosed mass stays the same}} \frac{G}{R^2} \tag{426}$$

We can rewrite this:

$$\rho_{\mathrm{ta}} = \frac{3H_{\mathrm{ta}}^2}{8\pi G}\xi \tag{427}$$

where $\xi$ is the overdensity parameter, which is the overdensity of the halo with respect to the background at turnaround ($\xi > 1$ for overdensities). Then using $\tau$ and $y$, we have:

$$\boxed{y'' = -\frac{\xi}{2y^2}} \tag{428}$$

with the boundary conditions

$$\begin{aligned} y'|_{x=1} &= 0 \\ y|_{x=0} &= 0 \end{aligned} \tag{429}$$

and we can solve the equations:

$$\begin{aligned} x' &= x^{-1/2} \\ y'' &= -\frac{3}{2y^2} \end{aligned} \tag{430}$$

Then we get an implicit solution for $x$:

$$x' = x^{-1/2} \Rightarrow \boxed{\tau = \frac{2}{3}x^{3/2}} \tag{431}$$

So

$$\begin{aligned} x &= \left(\frac{3}{2}\right)\tau^{2/3} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{2}{3}\left(\frac{3}{2}\right)^{2/3}\tau^{-1/3} = x^{-1/2} \end{aligned} \tag{432}$$

We also have

$$y' = \pm\sqrt{\xi}\sqrt{\frac{1}{y} - 1} \tag{433}$$

using the first boundary condition. We also use the + before turnaround and the − after.

Then

$$\begin{aligned}\frac{dy'}{d\tau} &= \pm\sqrt{\xi}y'\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(\frac{1}{y}-1\right)^{1/2}\right) \\ &= \pm\sqrt{\xi}y'\frac{1}{2}\left(\frac{1}{y}-1\right)^{-1/2}(-y^{-2}) \\ &= -\frac{\xi}{2y^2}\left(\pm\frac{1}{\sqrt{\xi}}y'\left(\frac{1}{y}-1\right)^{-1/2}\right) \\ &= -\frac{\xi}{2y^2}\underbrace{\left(y'\left(\pm\sqrt{\xi}\sqrt{\frac{1}{y}-1}\right)\right)^{-1}}_{=1}\end{aligned} \tag{434}$$

Integrating before turnaround and using the second boundary condition gives us an implicit solution for $y$:

$$\boxed{\tau = \frac{1}{\sqrt{\xi}}\left(\frac{1}{2}\arcsin(2y-1) - \sqrt{y-y^2} + \frac{\pi}{4}\right)}. \tag{435}$$

At turnaround:

$$\begin{aligned} &x = 1 = y, \tau = \frac{2}{3} \\ &\Rightarrow \frac{2}{3} = \frac{1}{\sqrt{\xi}}\left(\frac{1}{2}\underbrace{\arcsin(1)}_{\pi/2} + \frac{\pi}{4}\right) = \frac{1}{\sqrt{\xi}}\frac{\pi}{2} \\ &\Rightarrow \xi = \left(\frac{3\pi}{4}\right)^2 \end{aligned} \tag{436}$$

so we get the overdensity parameter $\xi$.

At collapse:
We assume symmetry, so we get collapse at $\tau = \frac{4}{3}$. Then

$$x_c = \left(\frac{3}{2}\right)^{2/3}\tau^{2/3} = \left(\frac{3}{2}\right)^{2/3}\left(\frac{4}{3}\right)^{2/3} = 4^{1/3} \tag{437}$$

**Collapse parameters**:

- Linearly extrapolated values:
  at early times, $y \ll 1$, so

  $$\tau \approx \frac{8}{9\pi}y^{3/2}\left(1 + \frac{3y}{10}\right). \tag{438}$$

  The overdensity inside the halo relative to the background is:

  $$\Delta = \left(\frac{x}{y}\right)\xi. \tag{439}$$

Note here that $\xi$ is the overdensity at turnaround, and we want to find $\Delta$ at collapse. The background density is proportional to $x^{-3}$, and the halo density is proportional to $y^{-3}$. We also know that $x = y = 1$ at turnaround, where $\Delta = \xi$. Now

$$\begin{aligned} \tau = \frac{2}{3}x^{3/2} \Rightarrow \left(\frac{2}{3}\right)x^{3/2} &\approx \frac{8}{9\pi}y^{3/2}\left(1+\frac{3y}{10}\right) \\ \Rightarrow \left(\frac{x}{y}\right)^{3/2} &= \frac{3}{2}\frac{8}{9\pi}\left(1+\frac{3y}{10}\right) \\ \Rightarrow \left(\frac{x}{y}\right)^{3} &= \underbrace{\left(\frac{4}{3\pi}\right)^2}_{=1/\xi}\underbrace{\left(1+\frac{3y}{10}\right)^2}_{\approx\left(1+\frac{3y}{5}\right)} \\ \Rightarrow \Delta = \left(\frac{x}{y}\right)^3 \xi &= 1+\frac{3y}{5} \end{aligned} \tag{440}$$

linear density contrast (assuming $y \ll 1$):

$$\delta = \Delta - 1 = \frac{3y}{5} \tag{441}$$

– The linearly extrapolated density contrast at turnaround is:

$$\delta_{\mathrm{ta}} = \frac{a_{\mathrm{ta}}}{a}\delta = \frac{\delta}{x} = \frac{3y}{5x} \tag{442}$$

since linear perturbations $\delta$ grow like the scale factor. Now

$$\frac{1}{x} = \left(\frac{3\tau}{2}\right)^{-2/3} \approx \left(\frac{3\pi}{4}\right)^{2/3}\frac{1}{y} \tag{443}$$

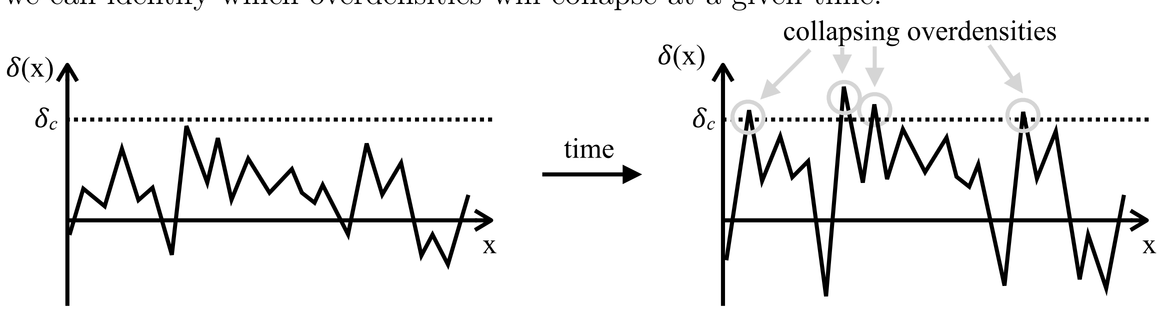
using the lowest order in $y$. We can then insert this into $\delta_{\mathrm{ta}}$ and get:

$$\delta_{\mathrm{ta}} = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} \approx 1.06 \tag{444}$$

– The linearly extrapolated density contrast at collapse is:

$$\delta_c = \frac{a_c}{a_{\mathrm{ta}}}\delta_{\mathrm{ta}} = x_c\delta_{\mathrm{ta}} = 4^{1/3}\delta_{\mathrm{ra}} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \approx 1.69 \tag{445}$$

So the halo can be considered collapsed when its density contrast expected from lineary theory has reached $\delta_c$. If we draw a density field as a function of one-dimensional space, we can identify which overdensities will collapse at a given time:



- Nonlinear values:
  We now look at the potential energy of a halo:

$$
\begin{aligned}
\text{at turnaround: } E &= V_{\mathrm{ta}} && \text{(no kinetic energy)} \\
\text{at collapse: } E &= T_c + V_c = \frac{1}{2}V_c && \text{(virial theorem: } 2T_c + V_c = 0) \\
\Rightarrow V_{\mathrm{ra}} &= E = \frac{1}{2}V_c \Rightarrow V_c = 2V_{\mathrm{ta}}
\end{aligned}
\tag{446}
$$

  Since potential energy is proportional to $\frac{1}{r}$ and $y = 1$ at turnaround, we know that $y = \frac{1}{2}$ at virialization. Then we get the overdensity at this time:

$$
\Delta_V = \left(\frac{x_c}{y}\right)^3 \xi = \left(\frac{4^{1/3}}{\frac{1}{2}}\right)^3 \xi = 32\xi = 32\left(\frac{3\pi}{4}\right)^2 = 18\pi^2 \approx 178 \tag{447}
$$

  A halo in virial equilibrium is expected to have a mean density of $\sim 178$ higher than the background. This is why masses and radii of halos are often quoted as $M_{200}$, which is the mass enclosed in a sphere of radius $R_{200}$ with an average density 200 times the mean or critical density of the Universe.

## 2.F Press-Schechter mass function

We want to know the halo mass function, i.e. the number density of a given mass of halos at a given redshift.

Analytic derivation:
We consider a halo of mass $M$. The characteristic length scale is then $R(M) = R$:

$$
\begin{aligned}
\frac{4\pi}{3}R^3\rho_c(z)\Omega_m(z) &= M \\
\Rightarrow R(M) &= \left(\frac{3M}{4\pi\rho_c(z)\Omega_m(z)}\right)^{1/3}
\end{aligned}
\tag{448}
$$

Halos of mass $M$ are then forming if the smoothed density field $\bar{\delta}$ crosses $\delta_c = 1.69$:

$$
\bar{\delta}(\vec{x}) = \int \mathrm{d}^3y\,\delta(\vec{x})W_R(|\vec{x} - \vec{y}|) \tag{449}
$$

where $W_R$ is the window function.

The variance on the scale $R(M)$ is:

$$
\sigma_R^2 = \frac{1}{2\pi}\int_0^\infty k^2\mathrm{d}k P(k)\hat{W}_R(k)\ . \tag{450}
$$

Inflation produces a Gaussian random field, so the probability of finding a smoothed density contrast $\bar{\delta}(\vec{x})$ at a given point in space $\vec{x}$ is:

$$
p(\bar{\delta}(\vec{x}), z) = \frac{1}{\sqrt{2\pi\sigma_R^2(z)}} e^{-\frac{\bar{\delta}^2(\vec{x})}{2\sigma_R^2(z)}} \tag{451}
$$

where $\sigma_R(z)$ is the linearly evolved $\sigma_R : \sigma_R(z) = \sigma_R D(z)$ for growth factor $D$.

The Press-Schechter idea is that the probability of finding the filtered density contrast at or above the linear density contrast for spherical collapse, $\bar{\delta} > \delta_c$, is equal to the fraction of volume filled with halos of mass $M$:

$$F(M,z) = \int_{\delta_c}^{\infty} d\bar{\delta} p(\bar{\delta}, z) = \frac{1}{2}\text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(z)}\right) . \tag{452}$$

The distribution of halos over mass $M$ is simply $\frac{\partial F(M,z)}{\partial M}$. To calculate this, we need:

$$\frac{\partial}{\partial M} - \frac{d\sigma_R(z)}{dM}\frac{\partial}{\partial \sigma_R(z)} = \frac{d\sigma_R}{dM}\frac{\partial}{\partial \sigma_R} . \tag{453}$$

Using $\frac{\mathrm{d}}{\mathrm{d}x}\text{erfc}(x) = -\frac{2}{\sqrt{\pi}}e^{-x^2}$, we get:

$$\begin{aligned}
\frac{\partial F(M,z)}{\partial M} &= \frac{\mathrm{d}\sigma_R}{\mathrm{d}M}\frac{\partial}{\partial \sigma_R}\left(\frac{1}{2}\text{erfc}\left(\frac{\delta_c}{\sqrt{2}D(z)\sigma_R}\right)\right) \\
&= \frac{\mathrm{d}\sigma_R}{\mathrm{d}M}\frac{1}{2}\left(-\frac{\delta_c}{\sqrt{2}D(z)\sigma_R^2}\right)\left(-\frac{2}{\sqrt{\pi}}e^{-\frac{\delta_c^2}{2\sigma_R^2 D^2(z)}}\right) \\
&= \frac{\mathrm{d}\sigma_R}{\mathrm{d}M}\frac{\delta_c}{\sqrt{2\pi}\sigma_R^2 D(z)}e^{-\frac{\delta_c^2}{2\sigma_R^2 D^2(z)}} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R D(z)}\frac{\mathrm{d}\ln(\sigma_R)}{\mathrm{d}M}e^{-\frac{\delta_c^2}{2\sigma_R^2 D^2(z)}}
\end{aligned} \tag{454}$$

so

$$\frac{\partial F(M,z)}{\partial M}\mathrm{d}M = \text{fraction of volume filled with halos of mass } [M, M+\mathrm{d}M]. \tag{455}$$

We must convert $\frac{\partial F}{\partial M}$ to an actual halo mass function. We convert to comoving number density by dividing by the mean volume $M/\rho_c(z)\Omega_M(z)$ occupied by mass $M$ halos:

$$\frac{\partial F(M,z)}{\partial M} = \frac{1}{\sqrt{2\pi}}\frac{\rho_c(z)\Omega_M(z)\delta_c}{\sigma_R D(z)}\frac{\mathrm{d}\ln(\sigma_R)}{\mathrm{d}M}e^{-\frac{\delta_c^2}{2\sigma_R^2 D^2(z)}}\frac{1}{M} \tag{456}$$

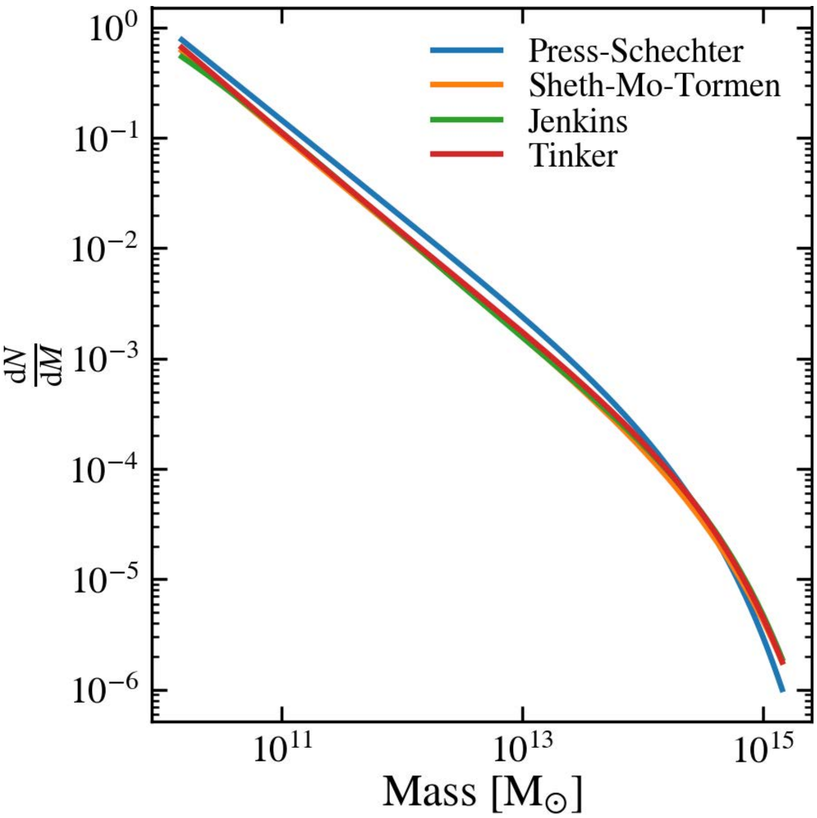
However, we need a fudge factor for the mass function to work. We require

$$\int_0^1 \mathrm{d}M\frac{\partial F(M,z)}{\partial M} = 1 \tag{457}$$

since $\frac{\partial F}{\partial M}$ is a volume fraction. But we get $\frac{1}{2}$ using $\frac{\partial F(M,z)}{\partial M}$ above! We therefore add a factor of two:

$$\boxed{N(M,z) = \sqrt{\frac{2}{\pi}}\frac{\rho_c(z)\Omega_M(z)\delta_c}{\sigma_R D(z)}\frac{\mathrm{d}\ln(\sigma_R)}{\mathrm{d}M}e^{-\frac{\delta_c^2}{2\sigma_R^2 D^2(z)}}\frac{1}{M}} . \tag{458}$$

See Problem Set 6 for a description of the Extended Press-Schechter formalism that explains the fudge factor.




Here we show the mass function for several theoretical models from Press and Schechter 1973, Sheth, Mo, and Tormen 2002, Jenkins et al. 2002, and Tinker et al. 2008. The lines are fairly similar, although the Press-Schechter deviates slightly more from the other models.