8.902 Fall 2023 - Problem Set # 5

Due Tuesday, November 14

Unless otherwise stated, assume $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_k = \Omega_r = 0$, and $H_0 = 71$ km/s/Mpc. Some problems will require you to perform numerical integration.

1 Mass of the Local Group: "Timing Argument"

The Andromeda Galaxy M31 and the Milky Way (MW) are the only two L^* galaxies in the Local Group. M31 also has the distinction of being the only major galaxy that is moving towards the MW, i.e. its light is blueshifted. After the Big Bang, the MW and M31 expanded away from each other, but eventually their mutual gravitational attraction halted this expansion and turned the system toward collapse.

After removing the rotation signature of the Sun about the Milky Way, our relative velocity toward M31 is measured at -121 km/s. Observations of Cepheid variables show that our current distance from M31 is approximately 700 kpc. According to Birkhoff's theorem, we may treat the MW–M31 system as a small closed universe, and parameterize the dynamics of this interaction as a cycloid:

$$r = A(1 - \cos\eta)$$
$$t = B(\eta - \sin\eta)$$

where A and B are numerical constants (*not* the Oort constants), r is the MW-M31 separation, t is the time since the Big Bang, and η runs from 0 to 2π.

A) Derive the following expression for the relative velocitiy between the MW and M31:

$$v = \frac{r}{t}\frac{\sin\eta(\eta - \sin\eta)}{(1 - \cos\eta)^2}$$

B) Starting from simple Newtonian mechanics, derive the folowing relation for M_T, the total mass of the MW-M31 system (with A and B defined above):

$$GM_T = \frac{A^3}{B^2} = \frac{r^3}{t^2}\frac{(\eta - \sin\eta)^2}{(1 - cos\eta)^3}$$

C) Since we are observing the distance and velocity today, the relevant timescale in the problem is the age of the Universe, 13.7×10^9 years. Using the expression derived in A) and the observed quantities, the current value of η.

D) Using the result from C), calculate the total mass of the MW-M31 system. The total luminosity of the system is roughly $3 \times 10^{10} L_\odot$. What is the mass-to-light ratio?

E) Since the MW has a flat rotation curve, we saw its mass scales as

$$M(< R) = \frac{v_c^2 R}{2G}$$

assuming the galaxy resides in a spherically symmetric dark matter halo. For a reasonable value of v_c, how far out in radius must the dakr matter extend to reach the mass of the MW calculated in D)? You may assume that the MW and M31 have the same mass for this part (in reality, M31 is somewhat larger). Are the dark matter haoes of the MW and M31 distinct or do they overlap?

F) What is the maximum historical separation of the MW-M31 system? How old was the Universe at that time? How long will it be before the MW collides with M31? Make a plot of r(t) from t=0 to the time of collision and indicate our present position in this diagram.

2 Supernova Ia Cosmology

Double weight in grading

The 2011 Nobel Prize in Physics was awarded to members of the teams that demonstrated accelerating expansion through observations of distant supernovae. From class, you should now understand that this is because Type Ia supernovae form an excellent standard candle against which to calibrate free parameters in the Luminosity Distance expression.

In this problem, we will examine how this is done in practice, using data from the Supernova Legacy Survey (SNLS) to fit cosmological parameters for a ΛCDM model. Although the full analysis uses Markov-Chain Monte Carlo calculations to fit a simultaneous solution to all parameters, we will divide the problem into two stages for simplicity. First, we will fit some "nuisance" parameters that describe the supernovae by assuming a flat universe. This means there is only one relevant cosmological parameter to fit (Ω_M, or $1 - \Omega_\Lambda$). After that, we will hold the nuisance parameters fixed and estimate probability contours for the cosmological parameters.

A) This problem will require you to numerical calculate luminosity distances for various different cosmologies. To start, plot D_L vs z out to $z = 5$ and a zoomed in version out to $z = 1$ relevant for this problem for the following cosmologies:

- $(\Omega_M, \Omega_\Lambda) = (1, 0)$ Flat, matter only
- $(\Omega_M, \Omega_\Lambda) = (0, 1)$ Flat, Λ only
- $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ Flat, current model
- $(\Omega_M, \Omega_\Lambda) = (1.2, 0)$ Closed, matter only
- $(\Omega_M, \Omega_\Lambda) = (0.3, 0)$ Open, matter only

Be sure to use the correct luminosity distance relation for flat, open, and closed Universes.

B) The class website contains a tarball of data files from the SNLS. Although the raw data are supernova lightcurves (Table 10), SNLS has provided lightcurve fits for each of their supernovae (Table 11). **All the data you need to use for this problem are in Table 11.** They have used two different lightcurve fitting methods: SALT2 and SiFTO, but we will only use SiFTO in this problem. Each lightcurve is characterized by a peak magnitude, shape parameter, and color parameter. The supernova's distance is then expressed in terms of a distance modulus of the following form:

$$\mu_i(\alpha, \beta, M) = m_i^* - M + \alpha s_i - \beta c_i$$

where μ_i, m_i^*, s_i, c_i are the distance modulus, peak magnitude, shape parameter, and color parameter for the i^{th} supernova; and the parameters α, β, M are "nuisance" parameters characterizing Type Ia supernovae that we will have to fit in addition to the cosmological parameters. The distance modulus calculated from the lightcurve in this way (i.e., by comparing the inferred absolute and measure apparent magnitudes) can be compared to the distance modulus μ_{cosmo} derived from the theoretical Luminosity Distance.

We fill fit for the nuisance parameters by minimizing the following **under the assumption of a flat Universe** ($\Omega_M + \Omega_\Lambda = 1$):

$$\chi^2 = \sum_{i \in SN} \frac{(\mu_i - \mu_{\text{cosmo}})^2}{\sigma_i^2 + \sigma_{\text{int}}^2}$$

where σ_{int}^2 is an intrinsic dispersion which you may assume to be 0.10 and σ_i^2 is the variance of μ_i. σ_i^2 can be calculate analytically, and the formula is given below for your convenience. The SNLS data table contains all necessary variances and covariances.

$$\begin{aligned}\sigma_i^2 &= \text{Var}(\mu_i) \\ &= \text{Var}(m_i^*) + \alpha^2 \text{Var}(s_i) + \beta^2 \text{Var}(c_i) + 2\alpha \text{Cov}(m_i^*, s_i) - 2\beta \text{Cov}(m_i^*, c_i) - 2\alpha\beta \text{Cov}(s_i, c_i)\end{aligned}$$

where we are assuming there is no error on the supernova redshift. Note that this formulate depends on α and β, and will bias your minimization towards very large α and β. To manage this problem, you should fit with the following algorithm

1. Initalize parameters to the following values: $\alpha = 1, \beta = 1, M = -20$. Also set some initial value for Ω_M (which μ_{cosmo} depends on)

2. Repeat until convergence: calculate $\sigma_i(\alpha, \beta)$, then fixing the σ_i, minimize χ^2 with respect to $(\alpha, \beta, M, \Omega_M)$.

To get full credit for this problem, turn in:

- Your best-fit values for $\alpha, \beta, M, \Omega_M$

- A plot showing μ_i and the final derived error σ_i for each supernova. Include a line showing μ_{cosmo} for your best-fit model, along with two additional line illustrating the effect of changing Ω_M by ± 0.2 (but keeping the cosmology flat)

- A comment on the sensitivy of SNIa data for specifcying Ω_M

C) After the discovery was first announced, there was considerable skepticism about Λ, much of it centered around the possibility of a hypothesize intergalactic dust. Explain the effect of intergalactic dust extinction on your plot from A) and why this might be a concern.

D) Now that you have fit for α, β, and M we will relax the assumption that the Universe is flat. Keeping those nuisance parameters fixed, calculate χ^2 over a grid of Ω_M and Ω_Λ. Make a contour plot of χ^2, centered on the minimum and encompassing the minimum +2.3, +6.2, and +11.8 (these are similar to 1,2, and 3σ confidence intervals for a 2-parameter model). Is a flat universe favored? You will note that the confidence ellipse is highly elongated, indicating a degeneracy between matter and dark energy terms. Explain in a few short sentences why this degeneracy exists, and why the ellipse has the directional orientation that it does in the $\Omega_M - \Omega_\Lambda$ plane. Other cosmological measurements (i.e., CMB) break this degeneracy.