

Problem Set #10
Due in class Thursday, December 4, 2001.

1. Zel'dovich Approximation

The Zel'dovich approximation is to write the trajectories of pressureless matter (dark matter, or baryons on scales larger than the Jeans length) as

$$\vec{x}(\vec{q}, \tau) = \vec{q} + D_+(\tau)\vec{\psi}(\vec{q}), \quad (1)$$

where $\vec{\psi}(\vec{q})$ is determined by the initial conditions and $D_+(\tau)$ is the zero-pressure linear growing mode growth factor.

- a) Consider a spherically symmetric density perturbation of the form

$$\delta(\vec{x}, \tau_i) = \begin{cases} \delta_i > 0, & |\vec{x}| < q_0 \\ 0, & |\vec{x}| > q_0 \end{cases} \quad (2)$$

Find the radial displacement field $\vec{\psi}(\vec{q})$ corresponding to this density field to lowest order in δ_i . Show that the Zel'dovich approximation gives $\delta(|\vec{x}| < q_0, \tau) \rightarrow \infty$ at some finite $\tau = \tau_c$. What is the corresponding prediction for $\delta(\tau_c)$ from linear perturbation theory? Does the Zel'dovich approximation agree with the exact solution of the spherical infall model from Problem 3 above?

- b) The Zel'dovich approximation is exact for plane-parallel perturbations for trajectories that have not intersected others. Show this by considering a one-dimensional density field $\rho(x, \tau)$ with corresponding displacement field $\psi_x(q_x)$ and gravitational potential $\phi(x, \tau)$ obeying the Poisson equation $\partial_x^2 \phi = 4\pi G a^2 (\rho - \bar{\rho})$. Hint: substitute the Zel'dovich approximation trajectories into the exact equation of motion $d^2 \vec{x} / d\tau^2 + (\dot{a}/a) d\vec{x} / d\tau = -\vec{\nabla} \phi$. Show that the $\vec{\nabla} \phi$ implied by this equation agrees with the solution of the Poisson equation assuming mass conservation.

2. Linear growing mode

Small-amplitude cosmological density fluctuations with comoving wavenumber obeying $H \ll k \ll k_J$ grow in amplitude according to the well-known damped, driven wave equation

$$\ddot{D} + \frac{\dot{a}}{a} \dot{D} = 4\pi G \bar{\rho}_m a^2 D \quad (3)$$

where a dot denotes a conformal time derivative.

- a) Suppose that the universe contains nonrelativistic matter, vacuum energy, and (possibly) curvature, but no other types of matter. By combining equation (3) with the Friedmann equation, show that one solution is given by $D_-(\tau) = H(\tau)$.
- b) Using this solution and the method of variation of parameters, find a quadrature solution for the growing mode, $D_+(\tau)$. (Hint: write $D_+ = D_-(\tau)f(\tau)$ and substitute into eq. 3. You should obtain $f(\tau)$ in the form of an integral.)
- c) Using the exact solution of $a(\tau)$ for an OCDM (matter-only, open) universe ($a \propto \cosh \eta - 1$ where $\eta \propto \tau$, from problem 2b of Problem Set 2), verify equation (15.31) of Peacock. Also show that for the Einstein-de Sitter model (flat $\Omega_m = 1$), $D_+ \propto a$.
- d) Now consider a flat Λ CDM model. Write the quadrature for $D_+(a)$ using expansion factor as the integration variable so that the quadrature is easy to evaluate numerically. Evaluate the growth suppression factor $g \equiv D_+(a = 1, \Omega_m = 0.35)/D_+(a = 1, \Omega_m = 1)$ numerically for the Λ CDM model and compare with equation (15.43) of Peacock. How accurate is Peacock's approximation?
- e) Suppose that the universe contains quintessence with equation of state $p = w\rho$, with w being (for simplicity here) a constant in the range $-1 < w < 0$. Show that the Peacock's equation (15.42) is invalid unless $w = -1$ or $w = -\frac{1}{3}$.

3. Simple model of nonlinear evolution

In hierarchical clustering models of cosmic structure formation, matter is clumped strongly on small scales and is smooth on large scales, with a transition mass M_{nl} that grows in time. One may define M_{nl} as the mean mass contained in a smoothing window large enough so that the filtered *linear* density fluctuation field has unit variance:

$$\sigma(M_{\text{nl}}) = 1 \quad \text{where} \quad \sigma^2(M) \equiv \int d^3k P(k)W^2(kR), \quad M = \frac{4\pi}{3}\bar{\rho}R^3. \quad (4)$$

The distribution of clump masses is given approximately by the Press-Schechter formula, Peacock equation (17.13). In this problem we use the gaussian window function $W(x) = \exp(-\frac{1}{2}x^2)$.

- a) Suppose that the power spectrum of $\delta\rho/\bar{\rho}$ in the linear regime may be approximated (for at least a useful range of k) as a power-law in k , $P(k, \tau) = D_+^2(\tau)Ak^n$. Determine $\sigma(M)$ in terms of the relevant constants. (Express the needed integral as a gamma function.)
- b) Check Peacock equation (17.14) for the low-mass slope of the Press-Schechter mass distribution.

- c) Today, $R_{\text{nl}} \approx 2.6 h^{-1}$ Mpc (using the gaussian window function). What is the corresponding M_{nl} in solar masses (in terms of Ω_m and h)? Now suppose that we take the characteristic mass of a galaxy to be $10^{11} h^{-1} M_\odot$. At what redshift was $\sigma = 1$ for this mass scale, i.e. what is the predicted redshift of galaxy formation? Evaluate your result for SCDM and Λ CDM using the standard parameters for these models with $n = 1$. (For Λ CDM you'll have to evaluate $D_+(a)$ by numerical integration.)

4. Gravitational radiation with a simple equation of state

Gravitational waves correspond to transverse-traceless metric perturbations h_{ij} with $\nabla_i h^i_j = h^i_i = 0$. The perturbations evolve according to the wave equation for a massless spin-2 field,

$$\left(\partial_\tau^2 + 2 \frac{\dot{a}}{a} \partial_\tau - \nabla^2 + 2K \right) h_{ij} = 8\pi G a^2 \Sigma_{ij,T} \quad (5)$$

where K is the spatial curvature constant and $\Sigma_{ij,T}$ is the transverse-traceless shear stress, the source for gravitational radiation. It vanishes in vacuum. Part a) of this problem is concerned with the evolution of the background spacetime; gravitational radiation is considered in the remainder. Assume that $\Omega \approx 1$ ($H^2 \gg a^{-2}|K|$) as is appropriate at high redshift even if $\Omega \neq 1$ today.

- a) Suppose that the unperturbed equation of state of the universe is $p = w\rho$ with $w = \text{constant}$ (e.g., $w = 0$ for a matter-dominated universe, $w = 1/3$ for a radiation-dominated universe, and $w = -1$ for a vacuum energy-dominated universe). Solve the Friedmann and energy conservation equations for the unperturbed Robertson-Walker spacetime to get $a(\tau)$. Show that for $w > -1/3$ the result is a power law of τ but for $w < -1/3$ the solution is a power of $\tau_\infty - \tau$. What is the physical interpretation of τ_∞ ? How does this relate to inflation?
- b) Show that gravitational radiation has two polarizations. (Hint: use the information given at the beginning of this problem.) Write down 3×3 matrices ϵ_{ij}^+ and ϵ_{ij}^\times corresponding to the two independent linear polarization states of plane gravitational waves traveling in the x_3 -direction. The general gravitational wave may then be written $h_+(\underline{k}, \tau)\epsilon_{ij}^+ + h_\times(\underline{k}, \tau)\epsilon_{ij}^\times$ summed over all different plane waves.
- c) For plane gravitational waves of comoving wavenumber k , show that the source-free gravitational wave equation has two solutions (for each polarization) given in terms of powers and Bessel functions of $k\tau$ (for $w > -1/3$) or $k(\tau_\infty - \tau)$ (for $w < -1/3$). Show that $h(k, \tau)$ is constant on scales much larger than the Hubble distance (neglecting the decaying mode). Thus, both density fluctuations and gravitational waves created during inflation can be

cosmologically important at late times when they reenter the Hubble distance.

- d) Show that the amplitude of gravitational waves decays as a^{-1} on scales small compared with the Hubble distance ($k\tau \gg 1$). Interpret this result physically in terms of the energy flux carried by gravitational waves. (A qualitative argument will suffice.) The decay of h once waves cross the Hubble length implies that tensor mode contributions to the CMB anisotropy are negligible for $l > 100$.