Massachusetts Institute of Technology
Department of Physics

Physics 8.942 Fall 2001

Problem Set #9
Due in class Tuesday, November 20, 2001.

1. **CDM density perturbation growth**

For wavelengths much less than the Hubble length ($kc \gg H$), the Fourier transform of the density perturbation for nonrelativistic matter, $\delta(\vec{k},\tau)$, obeys the well-known linear wave equation for small-amplitude, isentropic, longitudinal perturbations:

$$\ddot{\delta} + \frac{\dot{a}}{a}\,\dot{\delta} + \left(k^2 c_s^2 - 4\pi G\bar{\rho}a^2\right)\delta = 0 \tag{1}$$

where dots denote conformal time derivatives.

a) Consider a universe dominated by cold dark matter (CDM) with $c_s = 0$, but also including radiation. Ignoring the radiation perturbations — a useful approximation on scales smaller than the Hubble distance — equation (1) is valid for the CDM throughout the radiation- and matter-dominated eras (with $\bar{\rho}$ on the right-hand side being the mean density of the CDM). Using the solution of the Friedmann equation for $a(\tau)$ in a flat matter+radiation universe, $a(\tau) = a_1\tau + a_2\tau^2$ (be sure to express a_1 and a_2 in terms of H_0 and $a_{\rm eq} = [1+z_{\rm eq}]^{-1}$), show that the growing mode solution is $\delta \propto 1+(3/2)(a/a_{\rm eq})$. Interpret this result physically for $a < a_{\rm eq}$ as well as $a > a_{\rm eq}$.

b) Rewrite the wave equation given above in terms of the gravitational potential perturbation $\phi(\vec{k},\tau)$. Show that $\dot{\phi} = 0$ for growing-mode density perturbations in a $\Omega = 1$ CDM-dominated universe, implying that in real space, $\phi = \phi(\vec{x})$. What happens to ϕ if $\Omega \neq 1$ or $c_s \neq 0$? Now consider an isolated, gravitationally bound, relaxed system (e.g., the Milky Way galaxy), whose mass and proper size do not change with time. Assuming that the gravitational potential at the center is finite, does it change with time? Can one, therefore, estimate the initial gravitational potential fluctuations from the gravitational potential well depths of galaxies and clusters?

2. **Hot Dark Matter**

Suppose that the universe is closed by one flavor of massive neutrino with $\Omega_\nu = 1$. The neutrinos stream freely (without scattering) since decoupling at $T \approx 1\,\text{MeV}$. The proper momentum of a typical neutrino is $p = (1+z)\,k_{\rm B}T_{\nu 0}/c$ where $T_{\nu 0} = (4/11)^{1/3}\,T_0$ is the present neutrino "temperature."

a) Show that the proper distance travelled by a typical neutrino (the "free-streaming distance") is

$$L_{\rm fs} = a_0 c \int_0^{\tau_0} d\tau\, \alpha\,(1+z)\left[1+\alpha^2\,(1+z)^2\right]^{-1/2} \quad \text{where } \alpha \equiv \frac{k_{\rm B}T_{\nu 0}}{m_\nu c^2} \qquad (2)$$

and $z = z(\tau)$ through the usual relation $1 + z = a_0/a(\tau)$.

b) At what redshift does the massive neutrino become nonrelativistic? Assuming that this occurs before $z_{\rm eq}$, compute $z_{\rm eq}$ for an $\Omega = 1$ universe with two flavors of massless neutrinos and one flavor of massive nonrelativistic neutrino. (cf. Problem 1 of Problem Set 4.)

c) Using the exact $z(\tau)$ for a $K = 0$ matter+radiation universe with two flavors of massless neutrinos and one flavor of massive nonrelativistic neutrino, and with $h = 0.72$, compute the exact free-streaming distance in Mpc by numerically integrating equation (2). How doe the results scale with m_ν and h if $\alpha \ll a_{\rm eq}$? If massive neutrinos are the dark matter, free-streaming erases primordial fluctuations for wavelengths up to about $L_{\rm fs}$.

3. **Spherical infall model**

The nonlinear evolution of spherically symmetric perturbations of a Friedmann-Robertson-Walker (zero pressure) universe is most easily described by integrating the trajectories of spherical shells containing fixed enclosed mass M.

a) Show that the solution $R(t)$ for the proper radius of a sphere enclosing fixed mass is given parametrically by $R = A(1 - \cos\eta)$, $t = B(\eta - \sin\eta)$. What is the value of A^3/B^2?

b) By comparing $R(t)$ with the result for an unperturbed $\Omega = 1$ Einstein-de Sitter universe, obtain the exact nonlinear solution (in parametric form) for $\bar\delta(t) \equiv M/\bar M(t) - 1$, where $\bar M = (4\pi/3)\bar\rho R^3$. (Hint: Follow a shell of fixed M and ask how much mass it *would* enclose, $\bar M(t)$, if the density were uniform and equal to the critical density $\bar\rho$. Note that $\bar\delta$ is the volume-average of the density perturbation $\rho/\bar\rho - 1$.)

c) If the enclosed mass remains constant and the initial energy is negative, $R \to 0$ implying $\delta \to \infty$. Show that this occurs for growing-mode initial perturbations when linear perturbation theory would predict $\bar\delta = \delta_c \approx 1.69$. Find the exact expression for δ_c in terms of π and simple numbers.

4. **Zel'dovich Approximation**

The Zel'dovich approximation is to write the trajectories of pressureless matter (dark matter, or baryons on scales larger than the Jeans length) as

$$\vec x\,(\vec q, \tau) = \vec q + D_+(\tau)\vec\psi\,(\vec q)\ , \qquad (3)$$

where $\vec{\psi}(\vec{q})$ is determined by the initial conditions and $D_+(\tau)$ is the zero-pressure linear growing mode growth factor.

a) Consider a spherically symmetric density perturbation of the form

$$\delta(\vec{x},\tau_i) = \begin{cases} \delta_i > 0 \ , & |\vec{x}\,| < q_0 \\ 0 \ , & |\vec{x}\,| > q_0 \end{cases} \tag{4}$$

Find the radial displacement field $\vec{\psi}(\vec{q})$ corresponding to this density field to lowest order in δ_i. Show that the Zel'dovich approximation gives $\delta(|\vec{x}\,| < q_0, \tau) \to \infty$ at some finite $\tau = \tau_c$. What is the corresponding prediction for $\delta(\tau_c)$ from linear perturbation theory? Does the Zel'dovich approximation agree with the exact solution of the spherical infall model from Problem 3 above?

b) The Zel'dovich approximation is exact for plane-parallel perturbations for trajectories that have not intersected others. Show this by considering a one-dimensional density field $\rho(x,\tau)$ with corresponding displacement field $\psi_x(q_x)$ and gravitational potential $\phi(x,\tau)$ obeying the Poisson equation $\partial_x^2\phi = 4\pi G a^2(\rho - \bar{\rho})$. Hint: substitute the Zel'dovich approximation trajectories into the exact equation of motion $d^2\vec{x}/d\tau^2 + (\dot{a}/a)d\vec{x}/d\tau = -\vec{\nabla}\phi$. Show that the $\vec{\nabla}\phi$ implied by this equation agrees with the solution of the Poisson equation assuming mass conservation.