Introduction to Neural Computation

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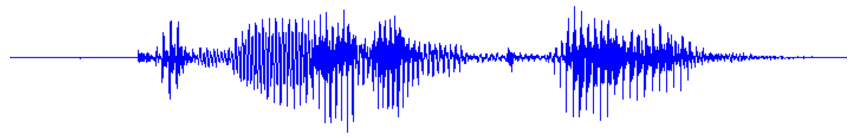
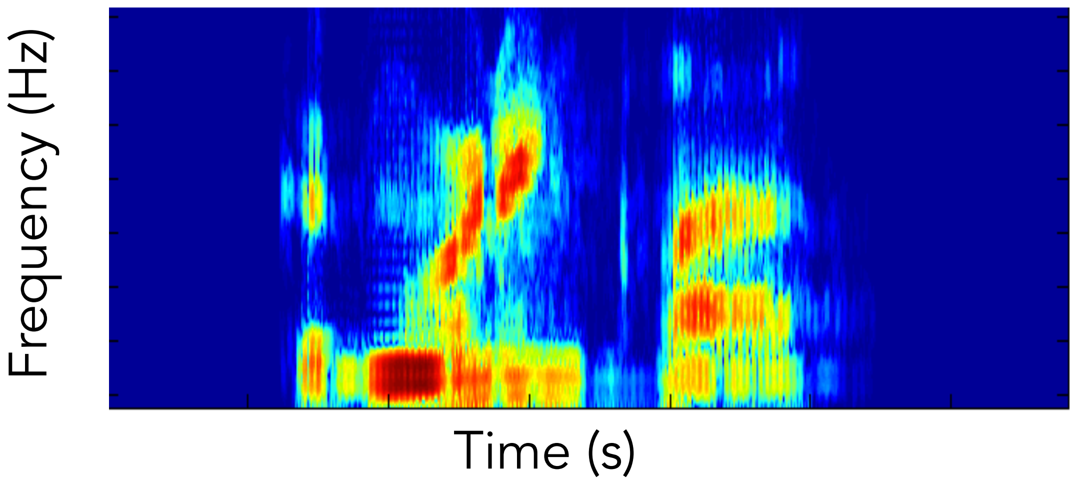
MIT BCS 9.40 — 2018

Lecture 11 - Spectral analysis I

Spectral Analysis





Game plan for Lectures 11, 12, and 13 —

Develop a powerful set of methods for understanding the temporal structure of signals

- Fourier series, Complex Fourier series, Fourier transform, Discrete Fourier transform (DFT), Power Spectrum
- Convolution Theorem
- Noise and Filtering
- Shannon-Nyquist Sampling Theorem
  - https://markusmeister.com/2018/03/20/death-of-the-sampling-theorem/
- Spectral Estimation
- Spectrograms
- Windowing, Tapers, and Time-Bandwidth Product
- Advanced Filtering Methods

Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum

Discrete Fourier transform

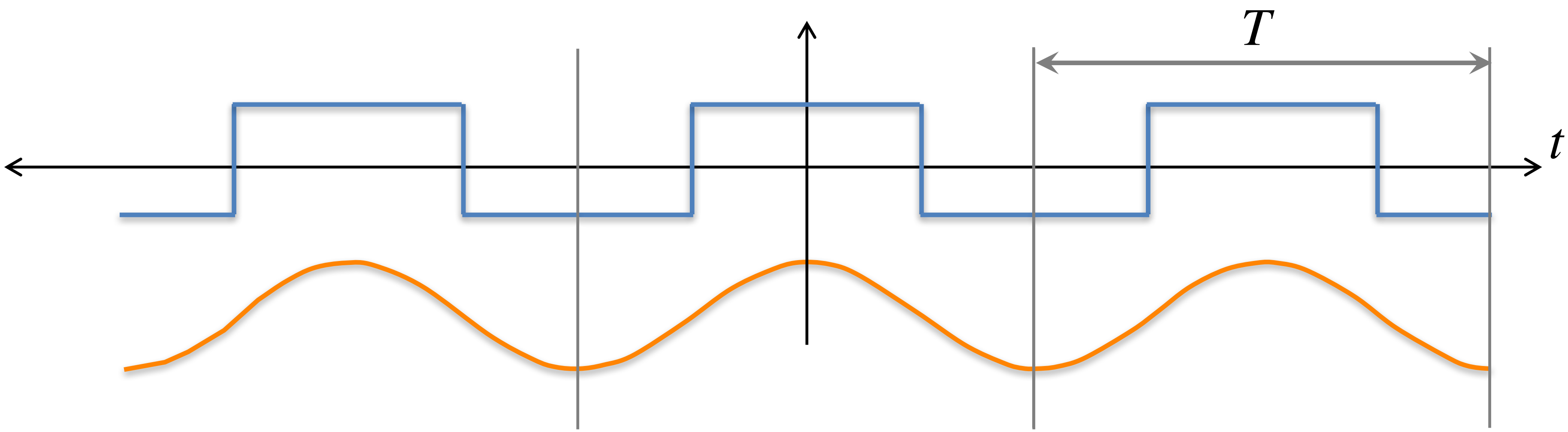
- Some code

WSpec.m | recordaudio.m | continuous_cos.m

```
%
N=2048;                  % number of samples in time
                         %
dt=.001;                 % sampling interval
Fs=1./dt;             % sampling frequency
time=dt*[-N/2:N/2-1];    % timebase
%
freq=20.;                % frequency of sine wave in Hz
y=cos(2*pi*freq*time);
%
%
yshft=circshift(y,[0,N/2]);     % First shift zero point from center to
                                    % first point in the array
ffty=fft(yshft, N)/N;           % Now compute the FFT

Y=circshift(ffty,[0,N/2]);          % Now shift the spectrum to put zero frequency
                                    % at the middle of the array
%
%Compute the vector of frequencies
df=Fs/N;
Fvec=df*[-N/2:N/2-1];
%
```

Fourier Series

- We can express any periodic function of time as sums of sine and cosine functions.
- Let's start with an even function that is periodic with a period T



We could approximate this square wave with a cosine wave of the same period T and amplitude.

$$a_1 \cos(2\pi f_0 t)$$

Oscillation frequency

$$f_0 = \frac{1}{T}$$

Cycles per second (Hz)

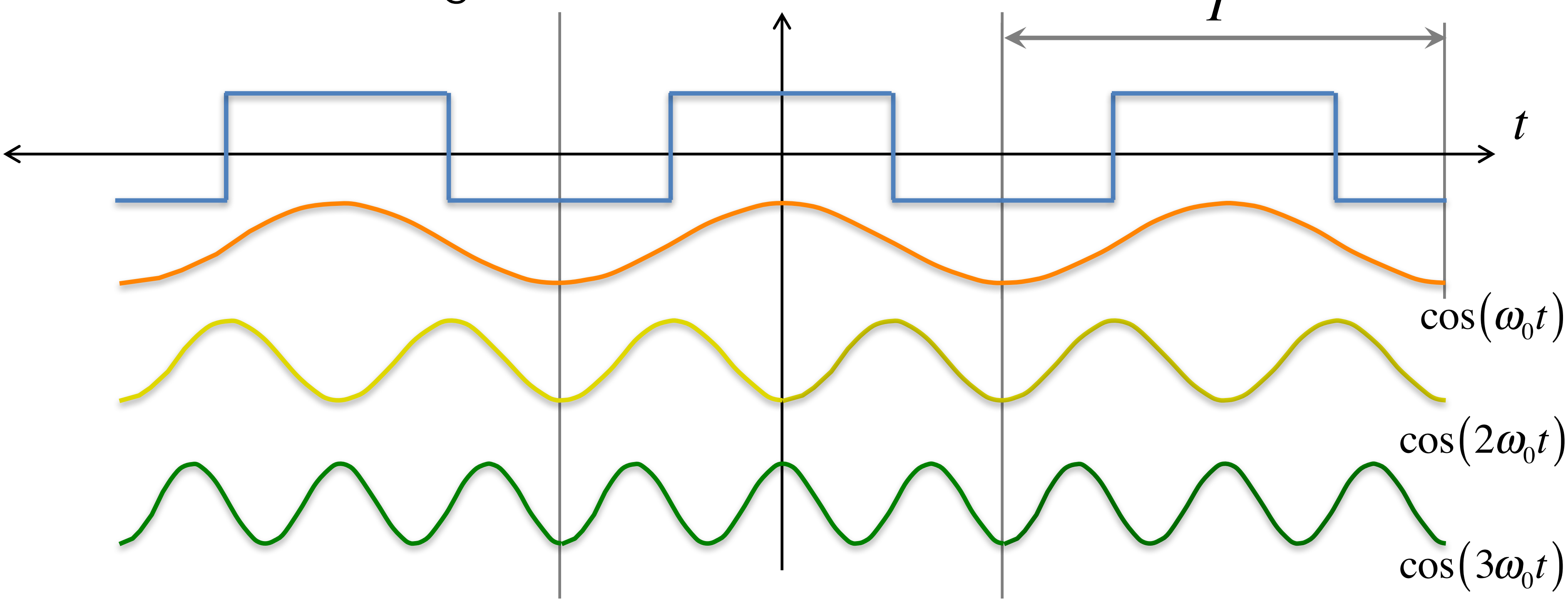
Angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

Radians per second

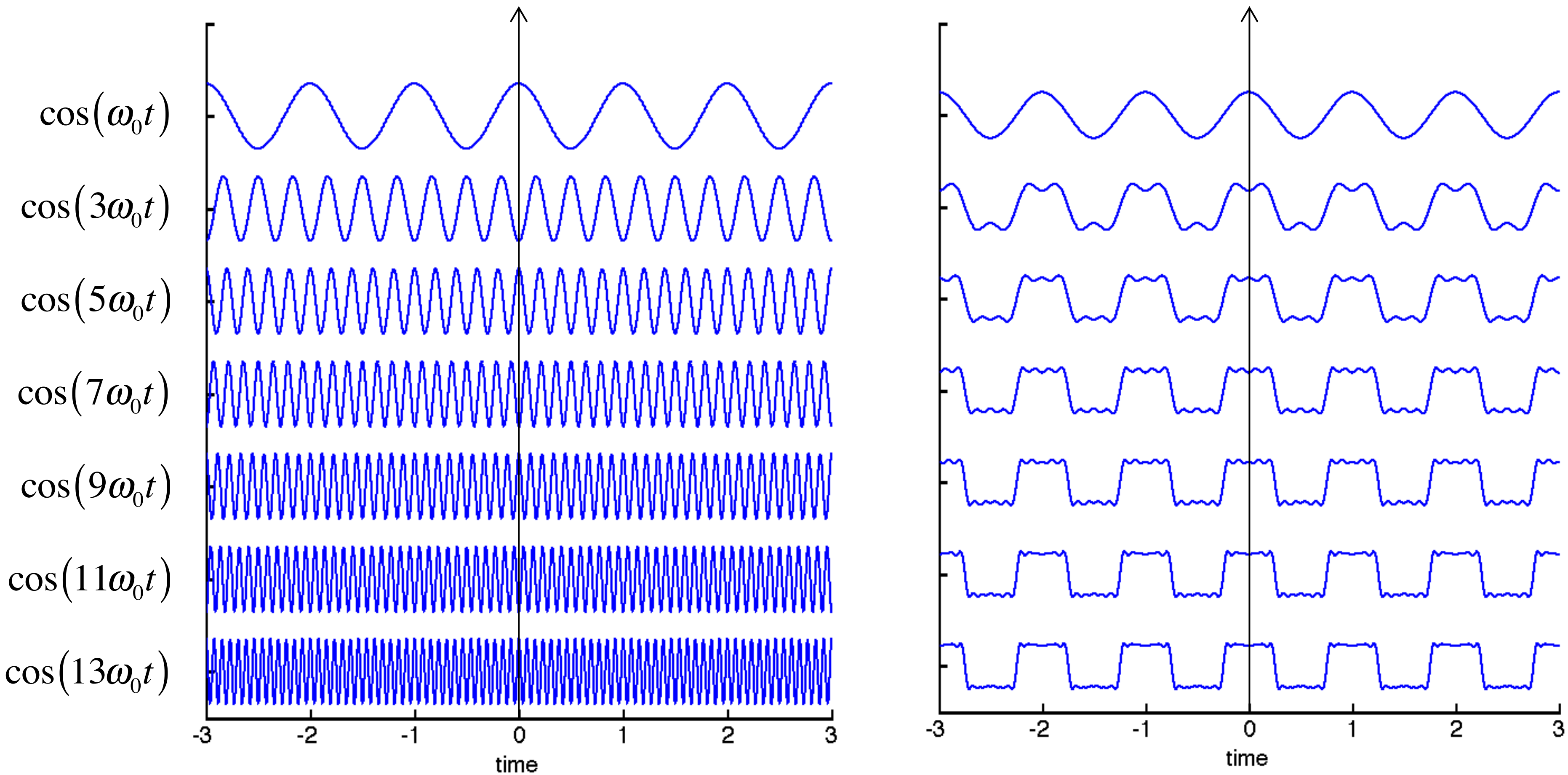
Fourier Series

- But we can get a better approximation if we add some more cosine waves to our original one…

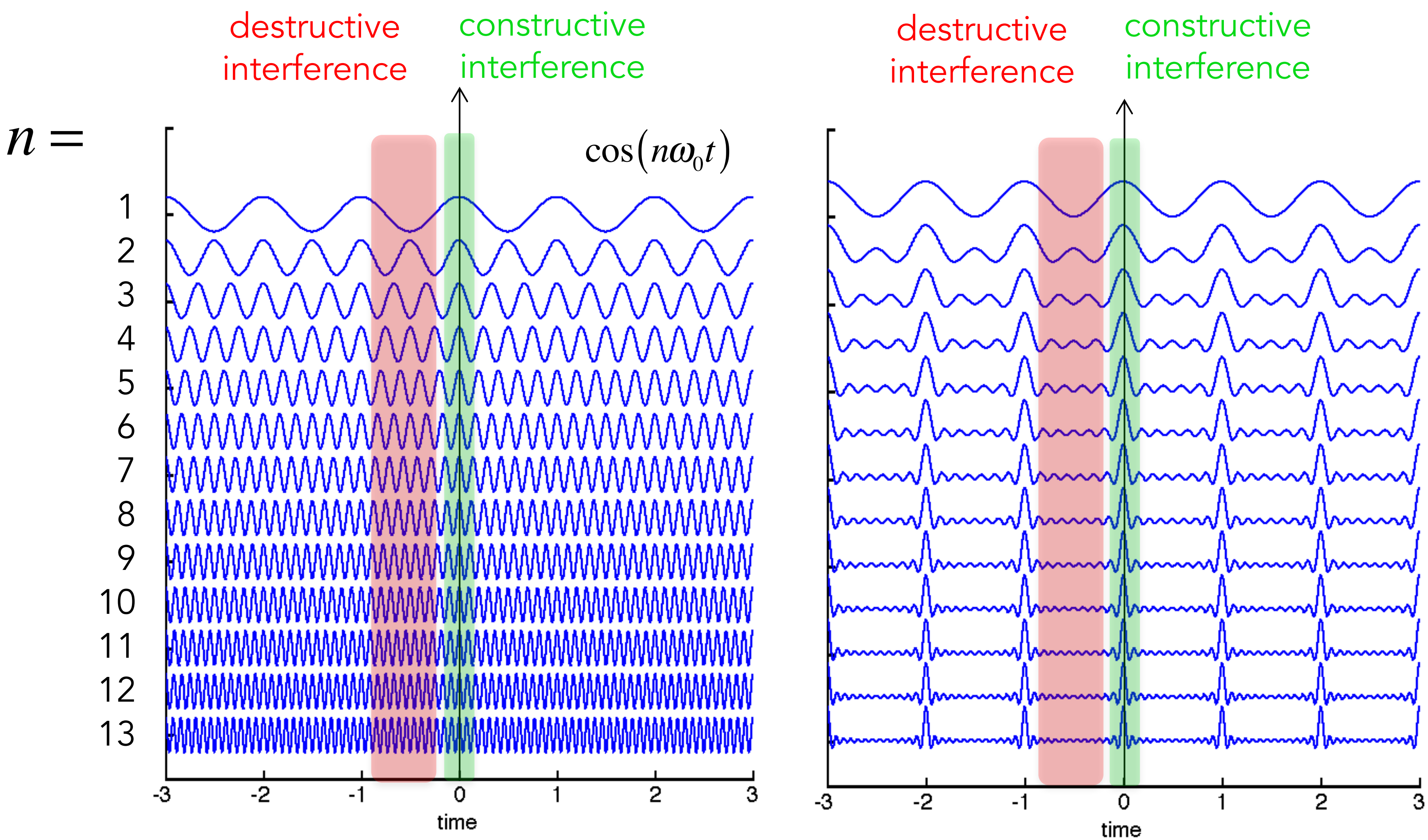


$$y(t) = a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$

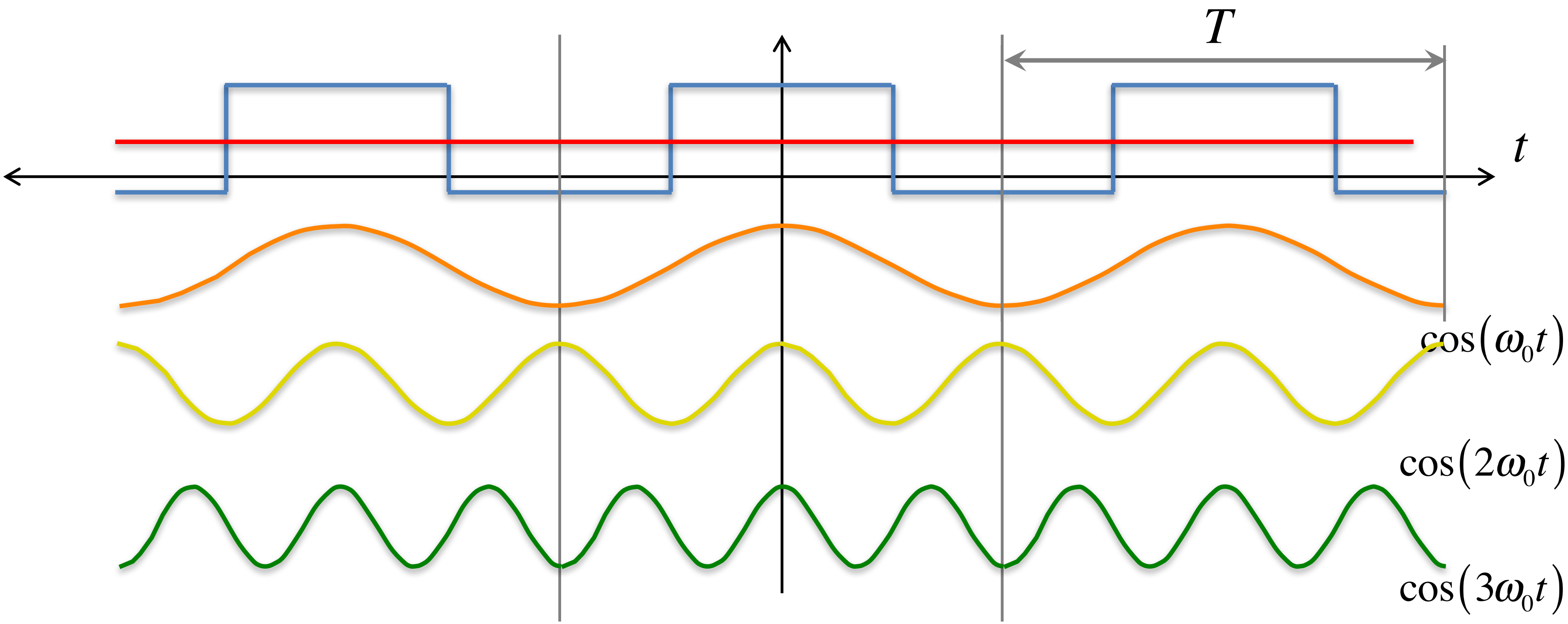
Fourier Series



Fourier Series



Fourier Series



$$y(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$

↑

DC term

$$y_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$$

How do we find the coefficients?

- The $a_0/2$ coefficient is just like the average of our function y(t).

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\,dt \qquad a_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(0\omega_0 t)\,dt$$

- The a_1 coefficient is just the **overlap** of our function y(t) with $\cos(\omega_0 t)$

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_0 t)\,dt$$

← Correlation!

- The a_2 coefficient is just the overlap of our function y(t) with $\cos(2\omega_0 t)$

$$a_2 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\omega_0 t)\,dt$$

- The a_n coefficient is just the overlap of our function y(t) with $\cos(n\omega_0 t)$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(n\omega_0 t)\,dt$$

How do we find the coefficients?

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\,dt \qquad a_1 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_0 t)\,dt \qquad a_2 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\omega_0 t)\,dt$$

Consider the following functions y(t):

$y(t) = 1$	$a_0 = 2$	$a_1 = 0$	$a_2 = 0$
$y(t) = \cos(\omega_0 t)$	$a_0 = 0$	$a_1 = 1$	$a_2 = 0$
$y(t) = \cos(2\omega_0 t)$	$a_0 = 0$	$a_1 = 0$	$a_2 = 1$

$$\int_{-T/2}^{T/2} \left[\cos(\omega_0 t)\right]^2 dt = \frac{T}{2}$$

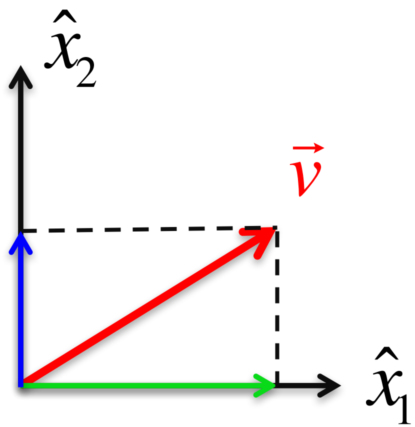
$$\int_{-T/2}^{T/2} \cos(\omega_0 t)\cos(2\omega_0 t)\,dt = 0$$

$$y(t) = \frac{a_0}{2} + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \ \ldots$$

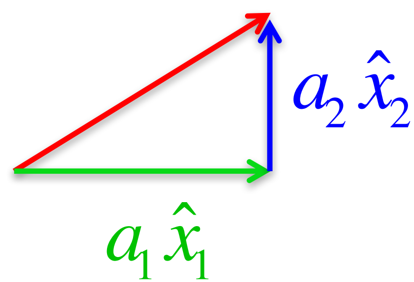
Fourier Series

- If a function has maximal overlap with one of our cosine functions, then it has zero overlap with all the others!
- We say that our set of cosine functions form an orthogonal basis set…

$$u_n(t) = \cos(n\omega_0 t)$$



$$\vec{v} = a_1\,\hat{x}_1 + a_2\,\hat{x}_2$$



$$\hat{x}_1 = [0\ ,1]$$

$$\hat{x}_2 = [1\ ,0]$$

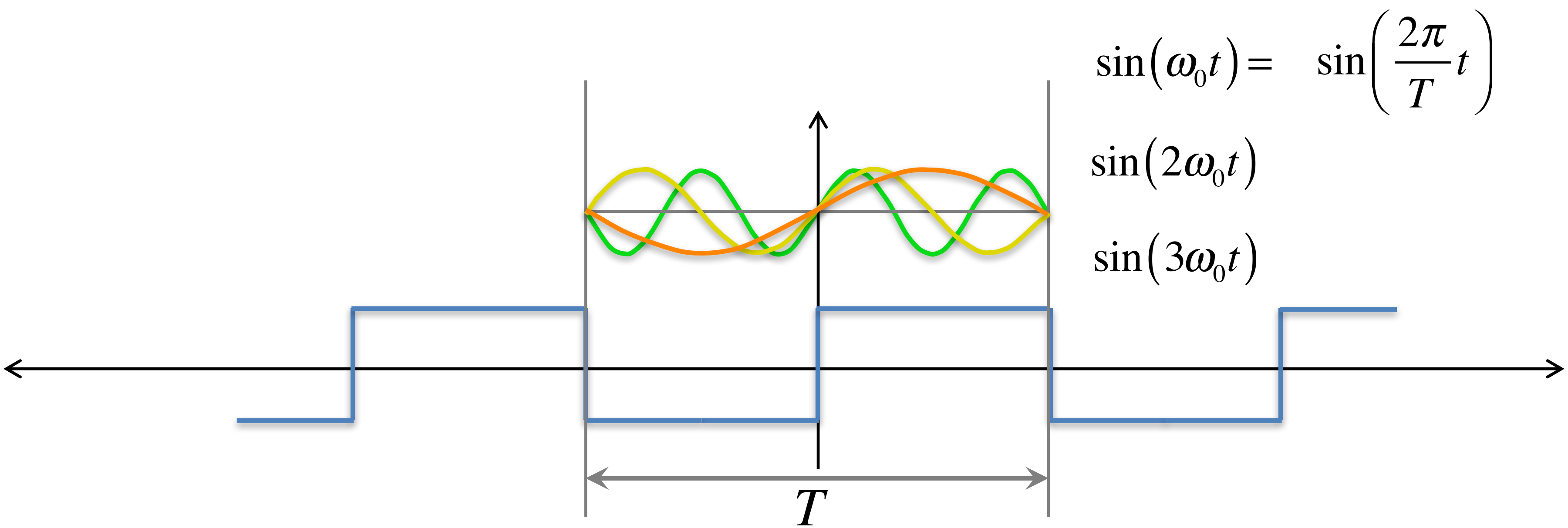
$$\vec{v} = [a_1\ ,a_2]$$

How do we find the coefficients a_1 and a_2?

$$a_1 = \vec{v}\cdot\hat{x}_1 = \sum_i v^i x_1^i \qquad a_2 = \vec{v}\cdot\hat{x}_2 = \sum_i v^i x_2^i$$

Fourier Series

- Now let's look an an odd (antisymmetric) function…



$$y_{odd}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \ldots$$

$$y_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Why is there no DC term here?

Fourier Series

- For an arbitrary function, we can write it down as the sum of a symmetric and an antisymmetric part.

$$y(t) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_0 t\right)}_{\text{symmetric}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(n\omega_0 t\right)}_{\text{antisymmetric}}$$

Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum

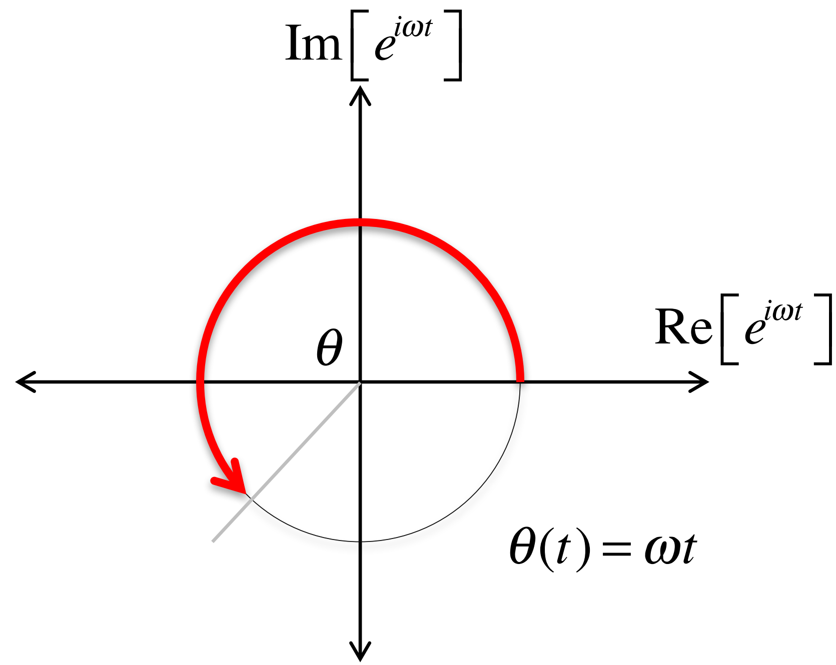
Complex Fourier Series

- We can express any periodic function of time as sums of complex exponentials.

Euler's formula

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$



Rewrite as follows…

$$\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin\omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) = -\frac{i}{2}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$\frac{1}{i} = -i$$

Fourier Series

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{in\omega t} + e^{-in\omega t}\right) + \sum_{n=1}^{\infty} \frac{-ib_n}{2}\left(e^{in\omega t} - e^{-in\omega t}\right)$$

$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t}$$

'DC' or 'constant' term

positive frequencies

negative frequencies

$$A_0 = \frac{a_0}{2} \qquad A_n = \frac{1}{2}(a_n - ib_n) \qquad A_{-n} = \frac{1}{2}(a_n + ib_n) \qquad A_n = (A_{-n})^*$$

complex conjugates

Complex Fourier Series

$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t}$$

- We can write this more compactly as follows:

$$= \sum_{n=0} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=-1}^{-\infty} A_n e^{in\omega_0 t}$$

$$For\ \ n = 0,$$

$$e^{in\omega_0 t} = e^0 = 1$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$

Complex Fourier Series

$$y(t) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)}_{\text{symmetric}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)}_{\text{antisymmetric}}$$

- We can replace the sine and cosines of the fourier series with a single sum of complex exponentials

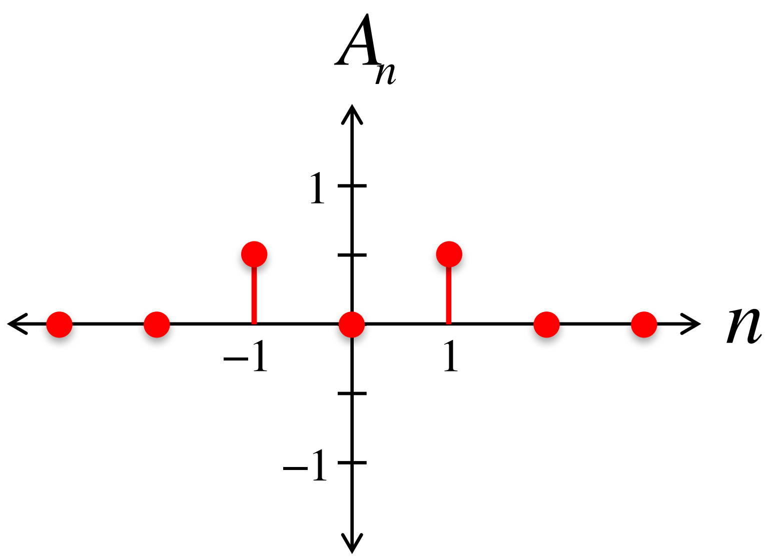
$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$

Complex Fourier Series

- Some examples…

$$A_{-1} = \frac{1}{2},\ A_0 = 0,\ A_1 = \frac{1}{2}$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$



$$\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$y(t) = \frac{1}{2}e^{-i\omega_0 t} + \frac{1}{2}e^{i\omega_0 t} = \frac{1}{2}\left(\cos\omega_0 t - i\sin\omega_0 t\right) + \frac{1}{2}\left(\cos\omega_0 t + i\sin\omega_0 t\right)$$

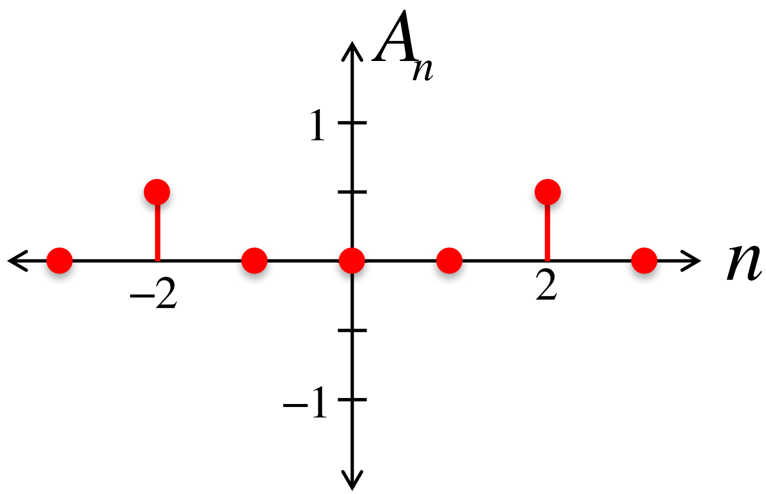
$$= \cos\omega_0 t$$

Complex Fourier Series

- Some examples…

$$A_{-2} = \frac{1}{2}\ ,\ A_0 = 0\ ,\ A_2 = \frac{1}{2}$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{i n \omega_0 t}$$



$$y(t) = \frac{1}{2} e^{-i2\omega_0 t} + \frac{1}{2} e^{i2\omega_0 t}$$

$$= \frac{1}{2}\left(\cos 2\omega_0 t - i \sin 2\omega_0 t\right) + \frac{1}{2}\left(\cos 2\omega_0 t + i \sin 2\omega_0 t\right)$$

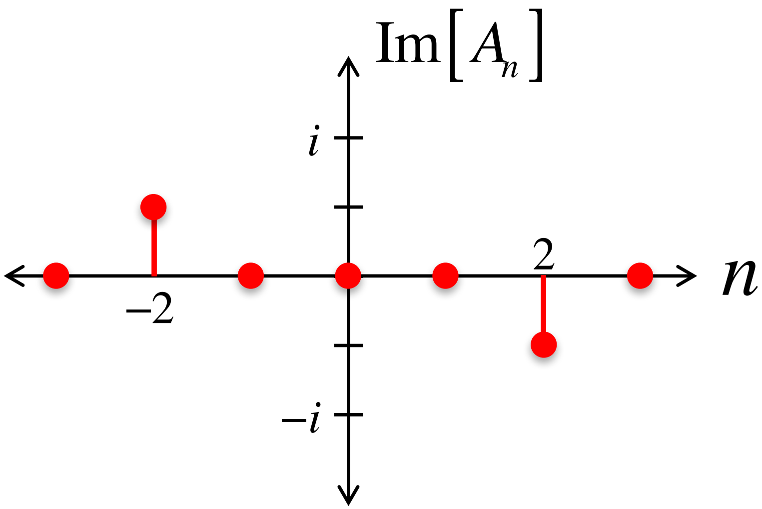
$$= \cos 2\omega_0 t$$

Complex Fourier Series

- Some examples…

$$A_{-2} = \frac{i}{2} \ , \ A_0 = 0 \ , \ A_2 = -\frac{i}{2}$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{i n \omega_0 t}$$



$$y(t) = \frac{i}{2} e^{-i2\omega_0 t} + \frac{-i}{2} e^{i2\omega_0 t}$$

$$= \frac{i}{2}\left(\cos 2\omega_0 t - i \sin 2\omega_0 t\right) + \frac{-i}{2}\left(\cos 2\omega_0 t + i \sin 2\omega_0 t\right) \quad = \quad \sin 2\omega_0 t$$

Complex Fourier Series

- The set of functions $e^{in\omega_0 t}$ form an orthogonal basis set over the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

- The A_0 coefficient is just the average of our function y(t).

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\,dt \qquad A_0 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-0 i\omega_0 t}\,dt$$

- The A_1 coefficient is just the overlap of our function y(t) with $e^{i\omega_0 t}$

$$A_1 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-i\omega_0 t}\,dt$$

In general

$$A_m = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-im\omega_0 t}\,dt$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$

Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform (I just want you to see this…)
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum

Fourier Transform

(for non-periodic functions)

$$A_m = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-im\omega_0 t}\,dt \qquad\qquad y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$

- We are going to do this by letting the period go to infinity!

$$T \to \infty \quad , \quad \omega_0 = \frac{2\pi}{T} \to 0 \qquad m\omega_0 \to \omega \qquad A_m \to Y(\omega)$$

Fourier Transform

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}\,dt$$

Inverse Fourier Transform

$$y(t) = \int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}\,\frac{d\omega}{2\pi}$$

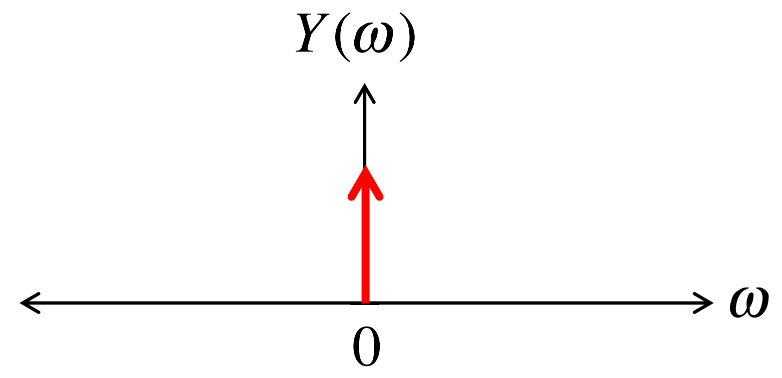
Fourier transform

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t}\, dt \qquad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t}\, d\omega$$

- Some examples…

$$y(t) = 1$$

$$Y(\omega) = 2\pi\delta(\omega)$$



$$y(t) = \int_{-\infty}^{\infty} \delta(\omega) e^{i\omega t}\, d\omega \quad = e^{i0t} \; = \; 1$$

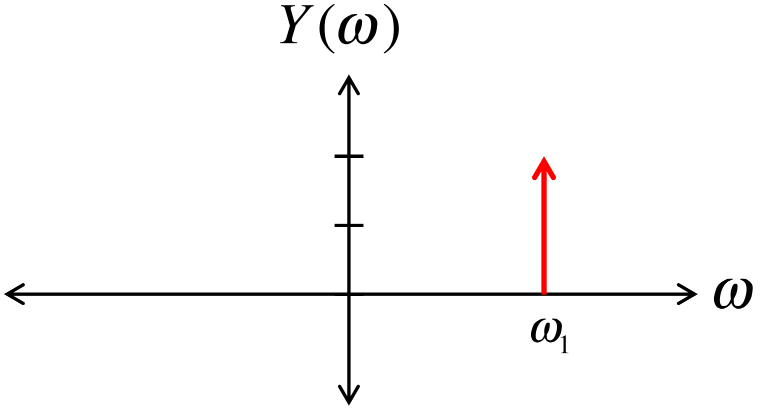
Fourier transform

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t}\, dt \qquad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t}\, d\omega$$

- Some examples…

$$y(t) = e^{i\omega_1 t}$$

$$Y(\omega) = 2\pi\,\delta(\omega - \omega_1)$$



$$y(t) = \int_{-\infty}^{\infty} \delta(\omega - \omega_1) e^{i\omega t}\, d\omega \;=\; e^{i\omega_1 t}$$

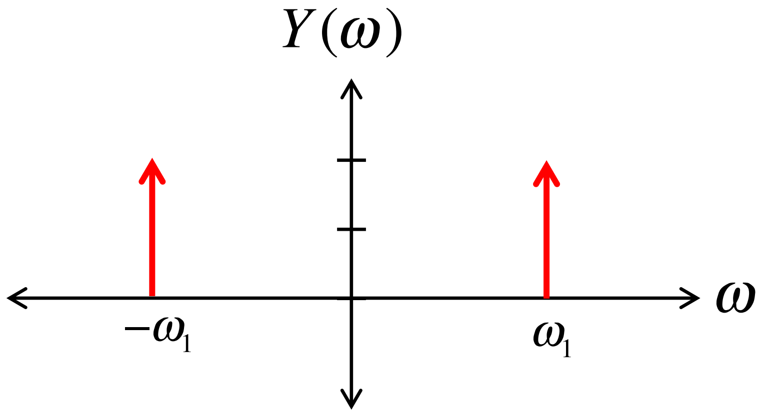
Fourier transform

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t}\, dt \qquad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t}\, d\omega$$

- Some examples…

$$Y(\omega) = \pi\left[\delta(\omega + \omega_1) + \delta(\omega - \omega_1)\right]$$

$$y(t) = \frac{1}{2} e^{-i\omega_1 t} + \frac{1}{2} e^{i\omega_1 t} = \cos\omega_1 t$$



Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum

Discrete Fourier transform

- Computing the FT and IFT is, in principle really slow
- You have to compute an integral for every value of ω you want in Y(ω).

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt$$

- It turns out there is a *super fast* computer algorithm called the Fast Fourier Transform (FFT).

First, let's go back to oscillation frequency f , rather than angular frequency ω:

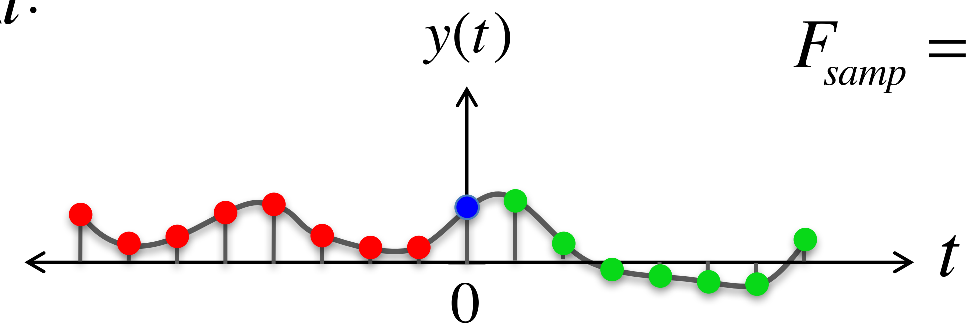
$$f = \omega / 2\pi$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi f t} \, dt \qquad y(t) = \int_{-\infty}^{\infty} Y(f) e^{i2\pi f t} \, df$$

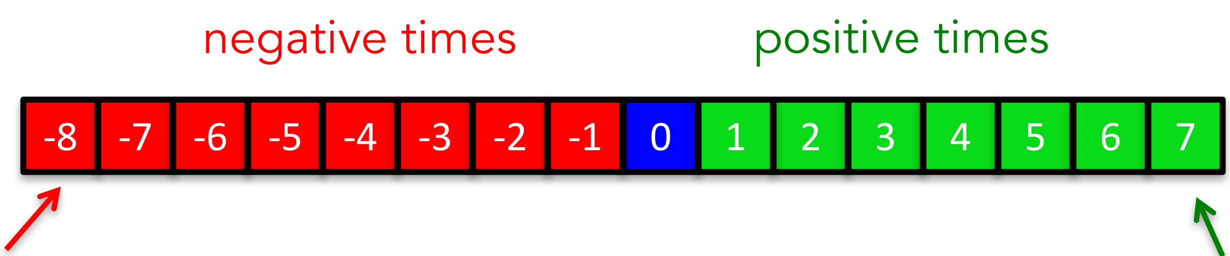
Discrete Fourier Transform

- Let's say we have a signal y(t) that is sampled at regular intervals Δt.

$$F_{samp} = 1/\Delta t$$



- Let's say we have N samples, and that N is an even number.



$$T_{\min} = -\frac{N}{2}\Delta t$$

$$T_{\max} = \left(\frac{N}{2} - 1\right)\Delta t$$

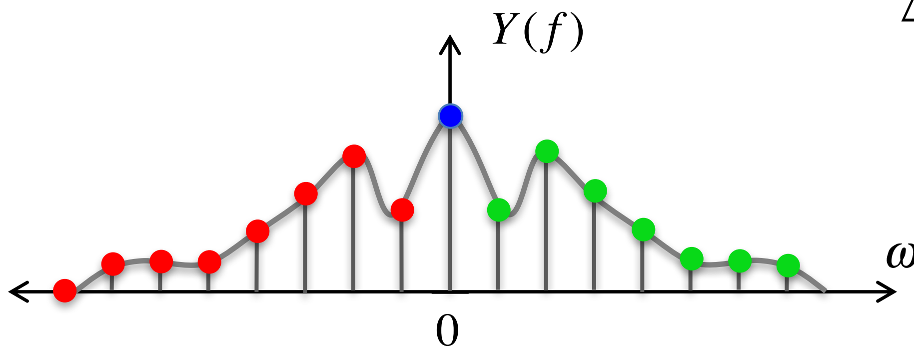
```
time=dt*[-N/2:N/2-1];   % time array
```

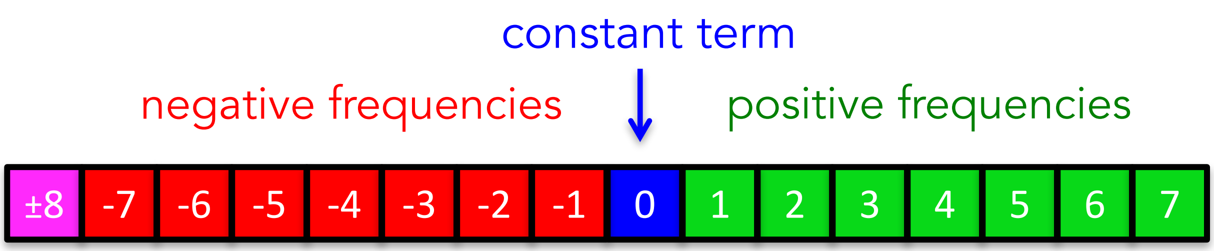
Discrete Fourier Transform

- The FFT algorithm returns a discrete Fourier transform that has N frequencies in frequency steps of Δf

$$\Delta f = \frac{F_{samp}}{N}$$

$$F_{nyquist} = \frac{F_{samp}}{2}$$





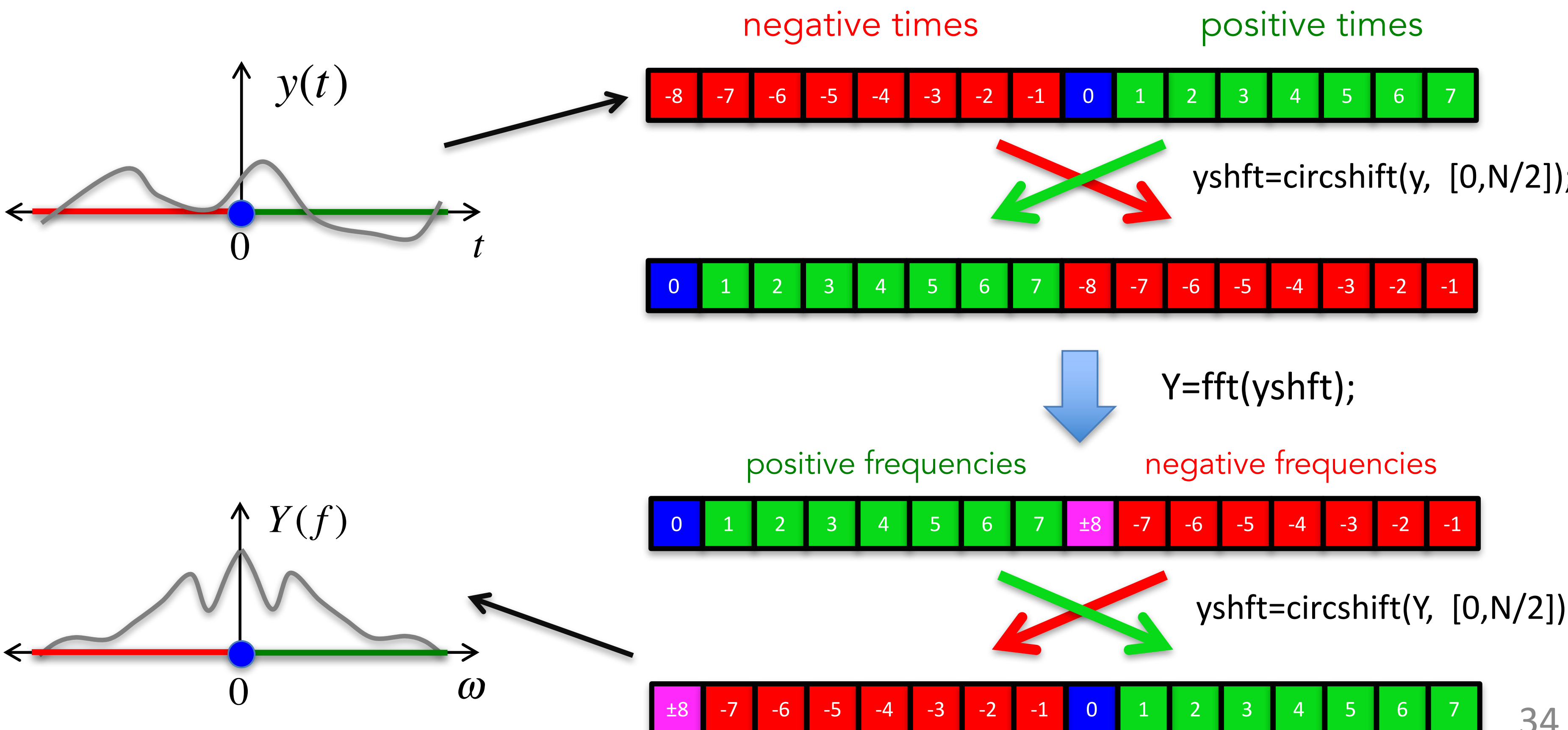
$$F_{bottom} = \pm \frac{N}{2} \Delta f$$

$$F_{top} = \left(\frac{N}{2} - 1 \right) \Delta f$$

```
freq=df*[-N/2:N/2-1];   % frequency array
```

Discrete Fourier transform

- One little trick… The FFT algorithm gets the time samples in a strange order, and returns the frequency samples in a strange order…



Discrete Fourier transform

- Some code

WSpec.m | recordaudio.m | continuous_cos.m

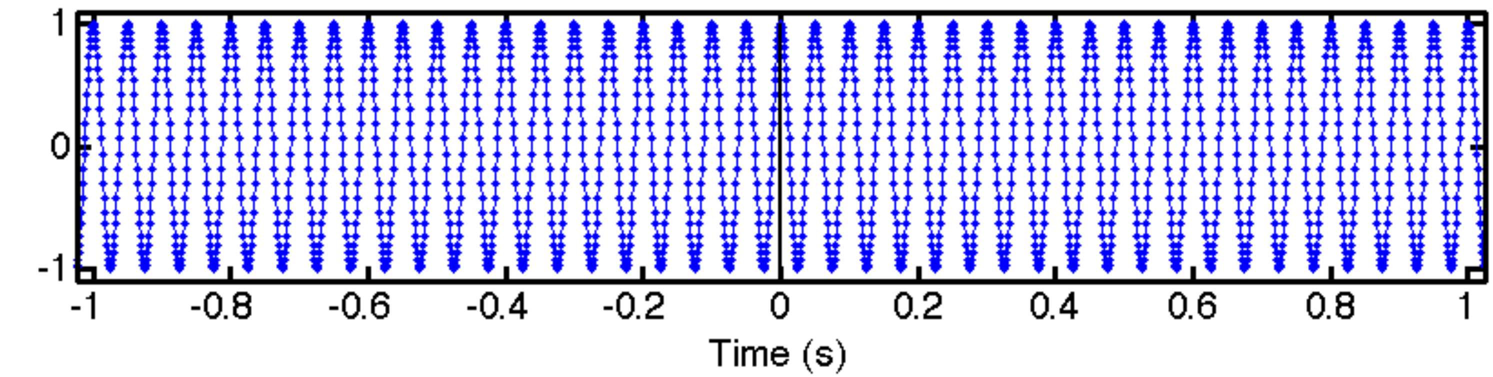
```
%
N=2048;                     % number of samples in time
                            %
dt=.001;                    % sampling interval
Fs=1./dt;               % sampling frequency
time=dt*[-N/2:N/2-1];   % timebase
%
freq=20.;                   % frequency of sine wave in Hz
y=cos(2*pi*freq*time);
%
%
yshft=circshift(y,[0,N/2]);     % First shift zero point from center to
                                    % first point in the array
ffty=fft(yshft, N)/N;          % Now compute the FFT

Y=circshift(ffty,[0,N/2]);          % Now shift the spectrum to put zero frequency
                                    % at the middle of the array
%
%Compute the vector of frequencies
df=Fs/N;
Fvec=df*[-N/2:N/2-1];
%
```

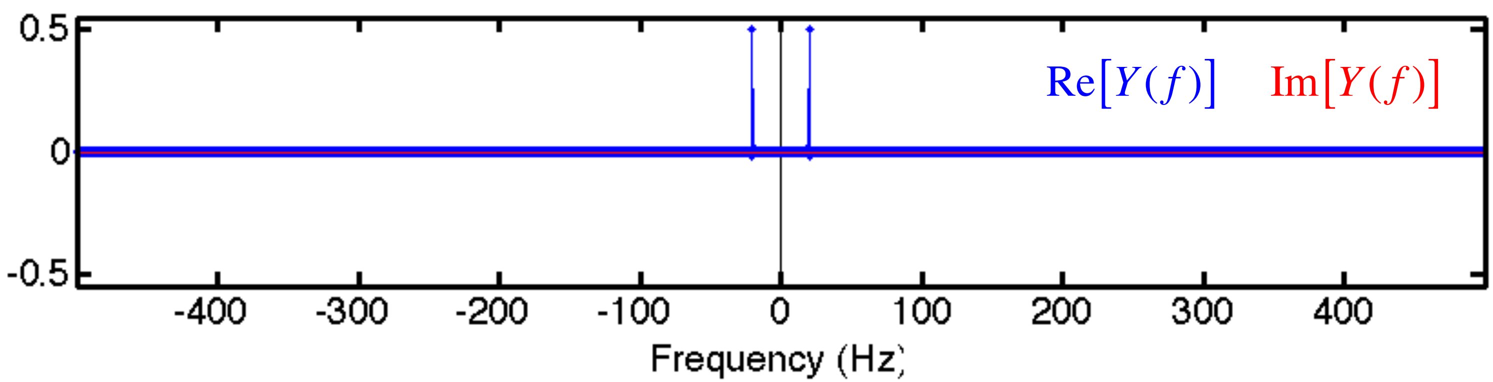
Discrete Fourier transform

- Some examples – sine and cosine

$$y(t) = \cos(2\pi f_0 t) \qquad f_0 = 20Hz$$

Continuous_cos.m



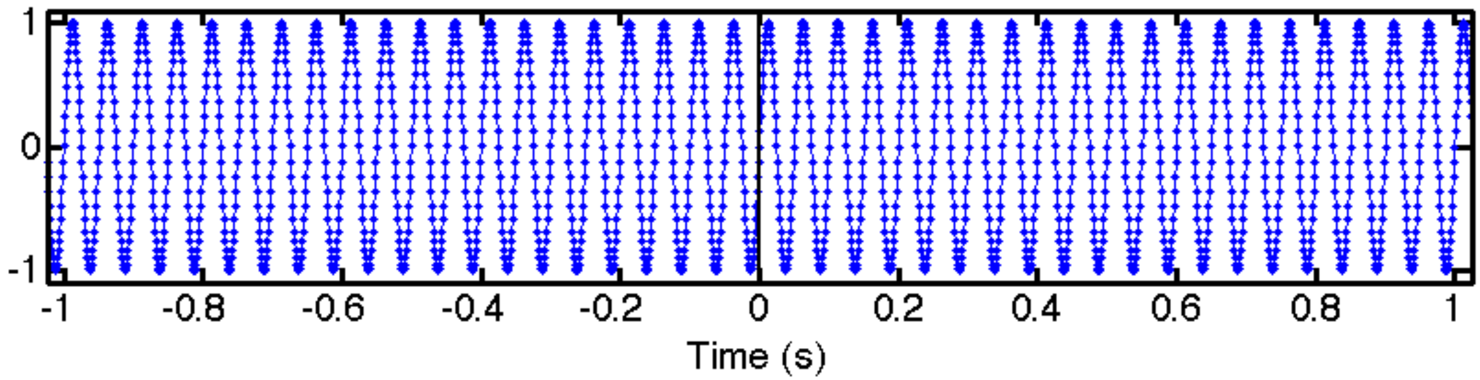


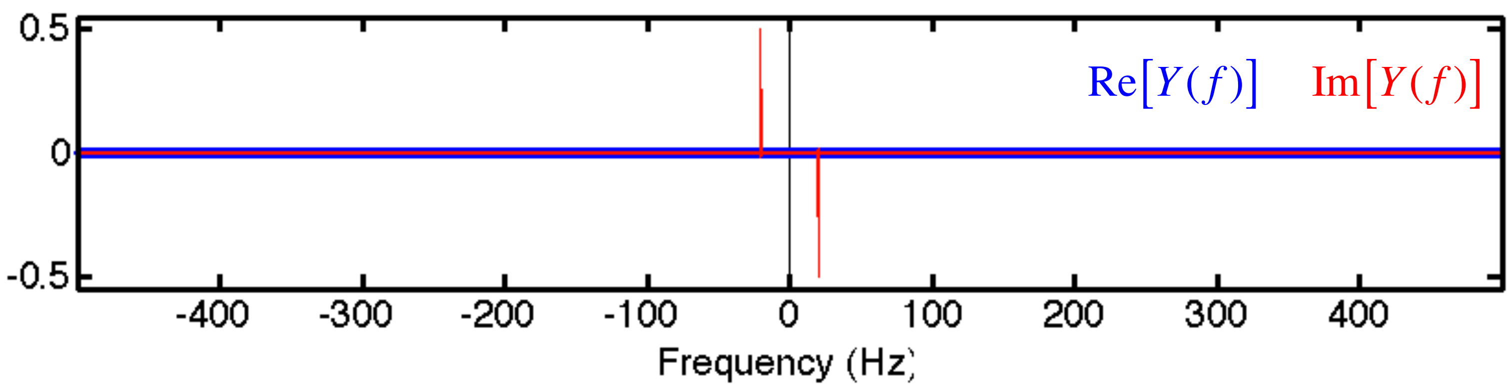
Discrete Fourier transform

- Some examples – sine and cosine

$$y(t) = \sin(2\pi f_0 t) \qquad f_0 = 20Hz$$

Continuous_sin.m





Learning Objectives for Lecture 11

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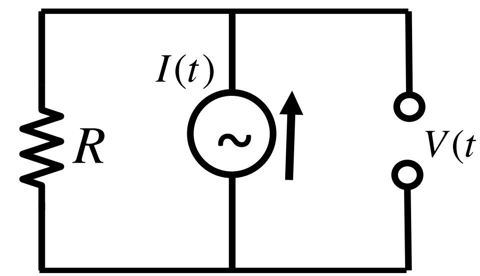
- Power spectrum

Introduce idea of 'Power'

- The electrical power dissipated in a resistor is given by

$$P(t) = I(t)V(t) = \frac{1}{R}V^2(t)$$



- If the voltage is just a single sine wave at frequency ω… $V(t) = \tilde{V}_\omega \cos(\omega t)$

$$V(t) = \tilde{V}_\omega \left[\frac{1}{2}e^{-i\omega t} + \frac{1}{2}e^{i\omega t} \right]$$

Then the average power from one frequency component is just given by the square magnitude of the F.T. at that frequency…

$$P(\omega) = \frac{1}{R}\left|\tilde{V}_\omega\right|^2 \left(\left|\frac{1}{2}e^{-i\omega t}\right|^2 + \left|\frac{1}{2}e^{i\omega t}\right|^2 \right) = \frac{1}{R}\left|\tilde{V}_\omega\right|^2 \left(\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 \right) = \frac{1}{R}\frac{\left|\tilde{V}_\omega\right|^2}{2}$$

Parseval's Theorem and Power

- The power in each frequency component independently contributes

$$E = \int_{-\infty}^{\infty} P(t)\,dt \qquad = \frac{1}{R}\int_{-\infty}^{\infty} [V(t)]^2\,dt$$

Parseval's Theorem says that

Power spectrum

$$\int_{-\infty}^{\infty} [V(t)]^2\,dt = \int_{-\infty}^{\infty} \left|\tilde{V}(\omega)\right|^2 \frac{d\omega}{2\pi}$$

Thus, each frequency component independently contributes to the power in the signal.

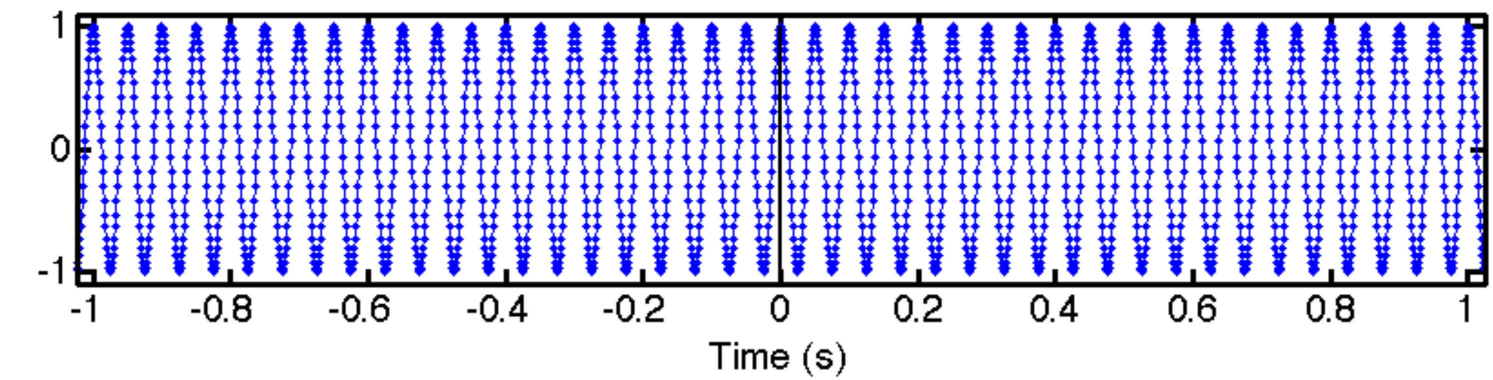
It also says that the total variance in the time domain signal is the same as the total variance in the frequency domain signal!

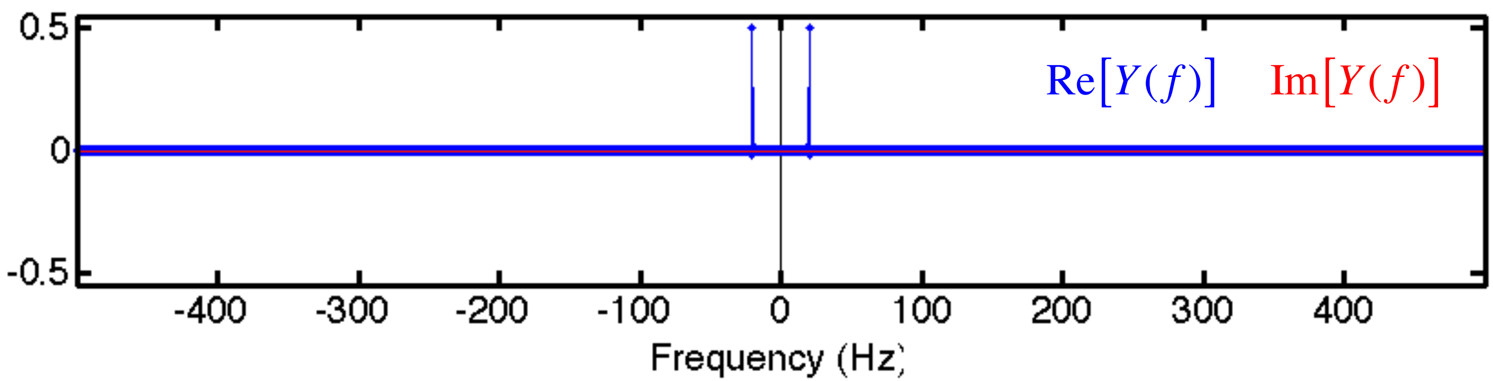
Discrete Fourier transform

- Some examples – sine and cosine

$$y(t) = \cos(2\pi f_0 t) \qquad f_0 = 20Hz$$

Continuous_cos.m

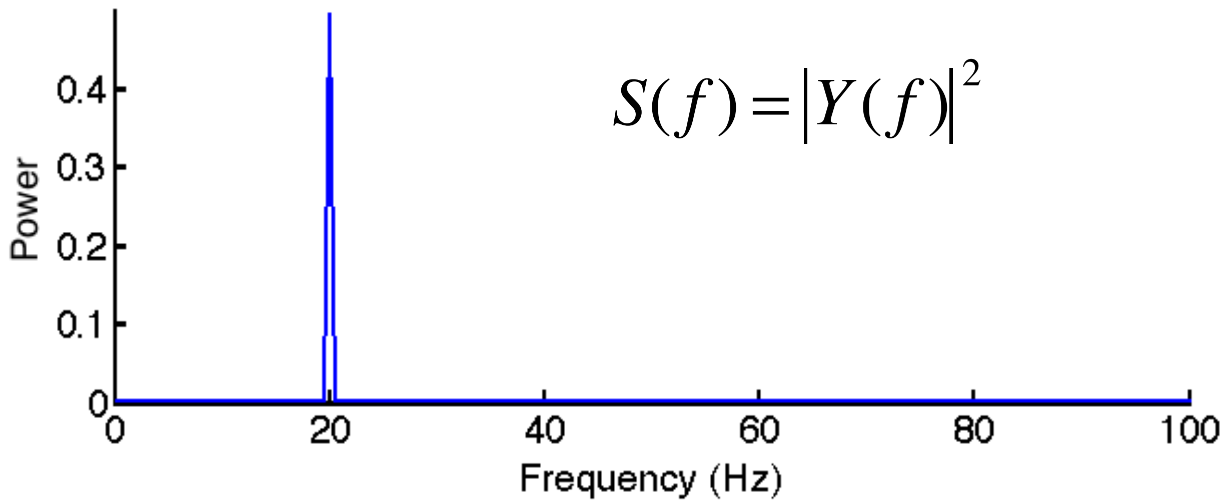




Discrete Fourier transform

- Power spectrum of sine and cosine

Continuous_sin.m



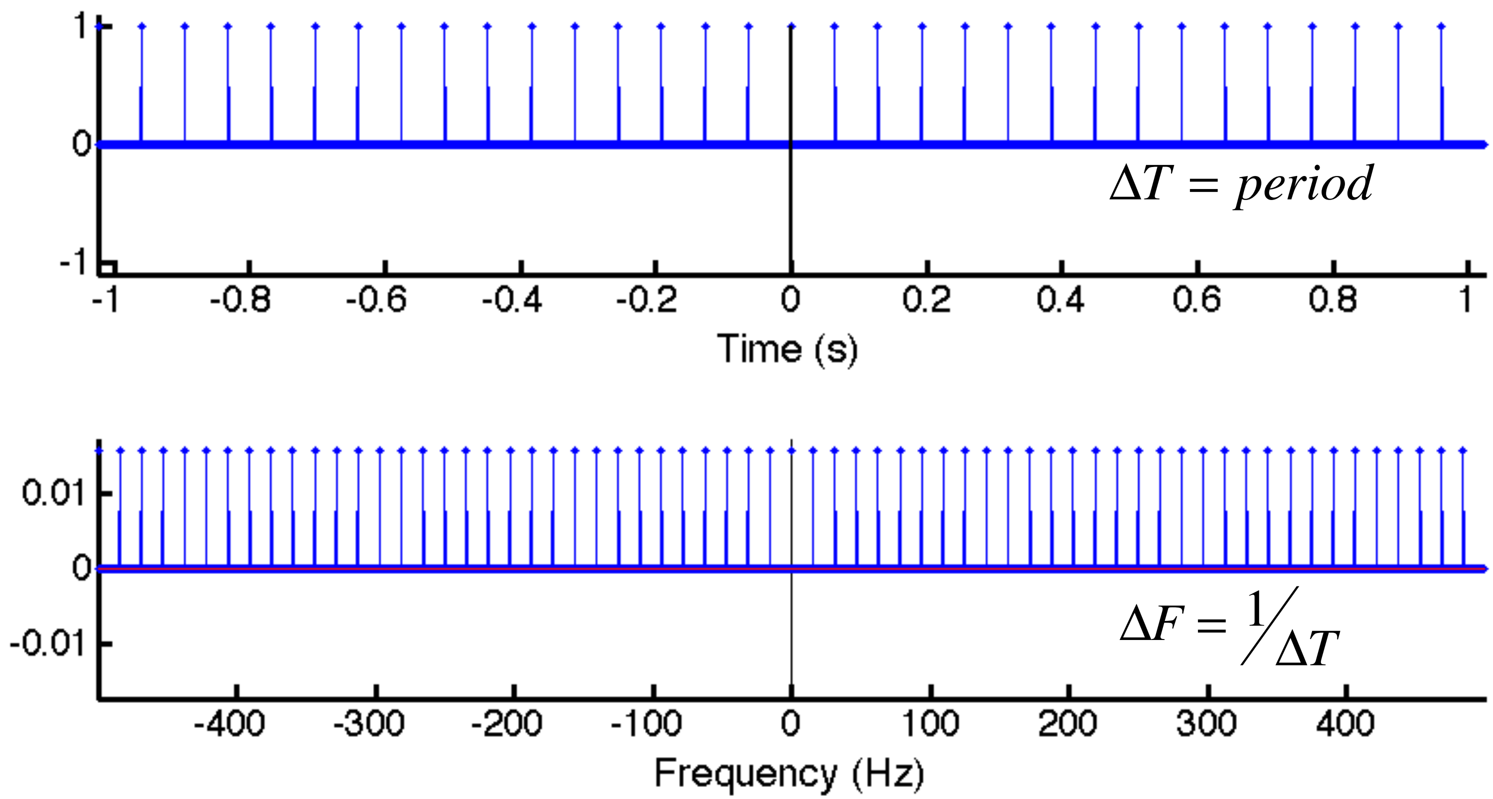
$$S(f) = |Y(f)|^2$$

For real signals, the power spectrum is symmetric, so only need to plot for positive frequencies!

Discrete Fourier transform

- Some examples – train of delta functions

deltafn_train.m

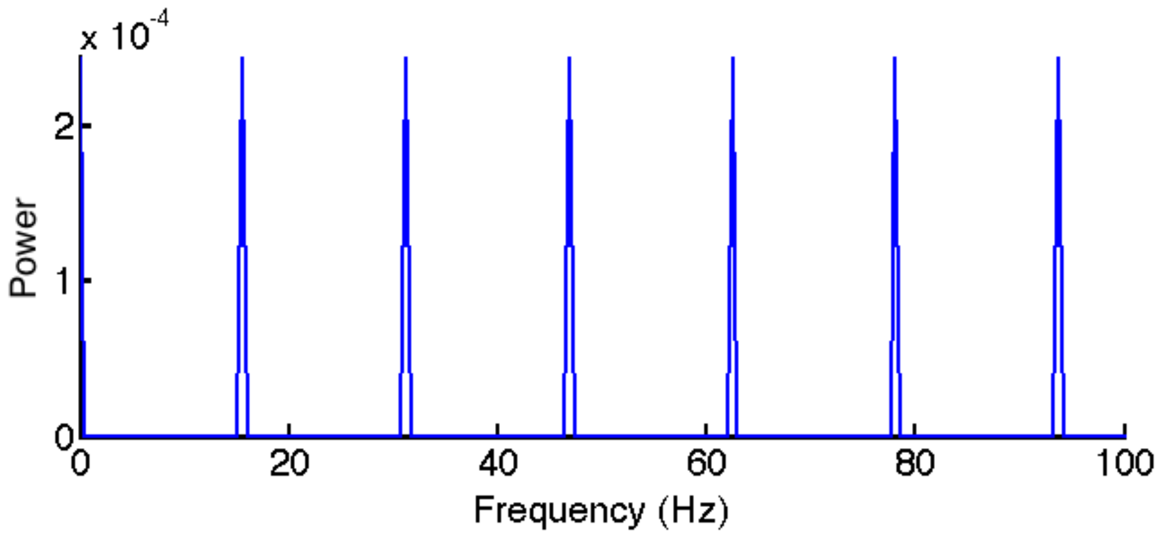


Discrete Fourier transform

- Power spectrum– train of delta functions

$$S(f) = |Y(f)|^2$$

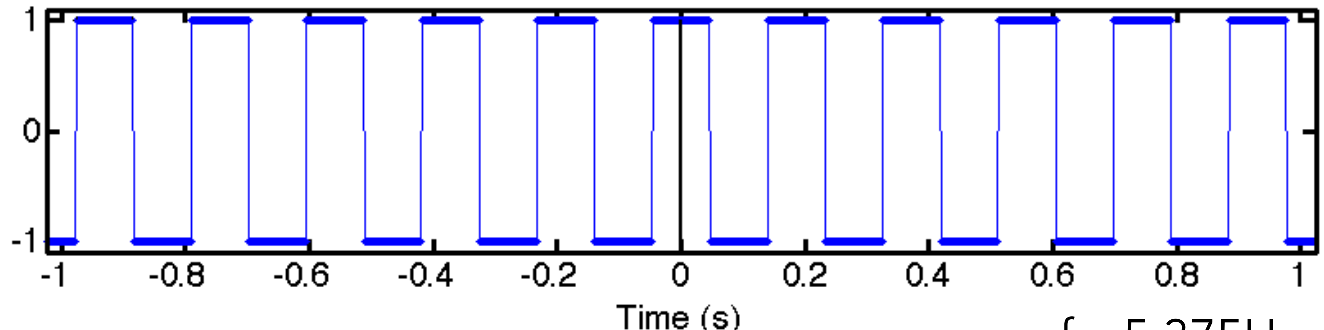
deltafn_train.m



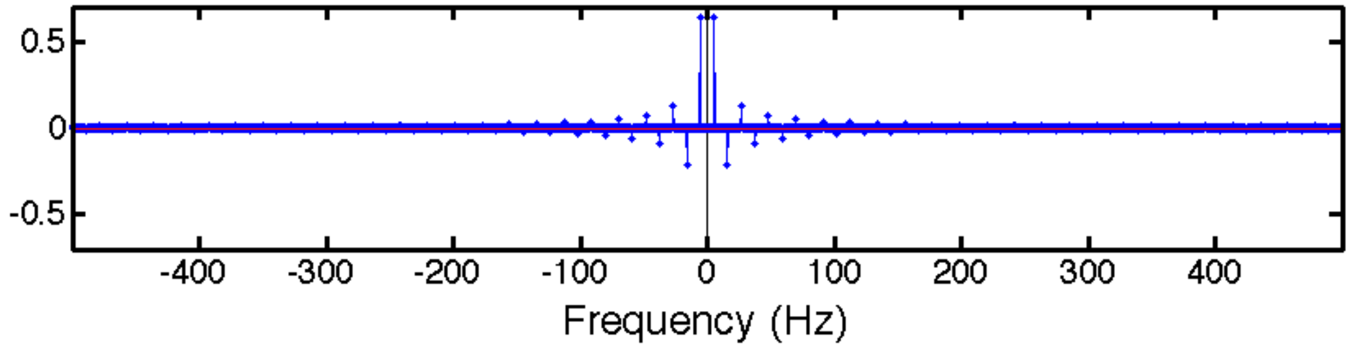
Discrete Fourier transform

- Some examples – square waves

Continuous_square.m



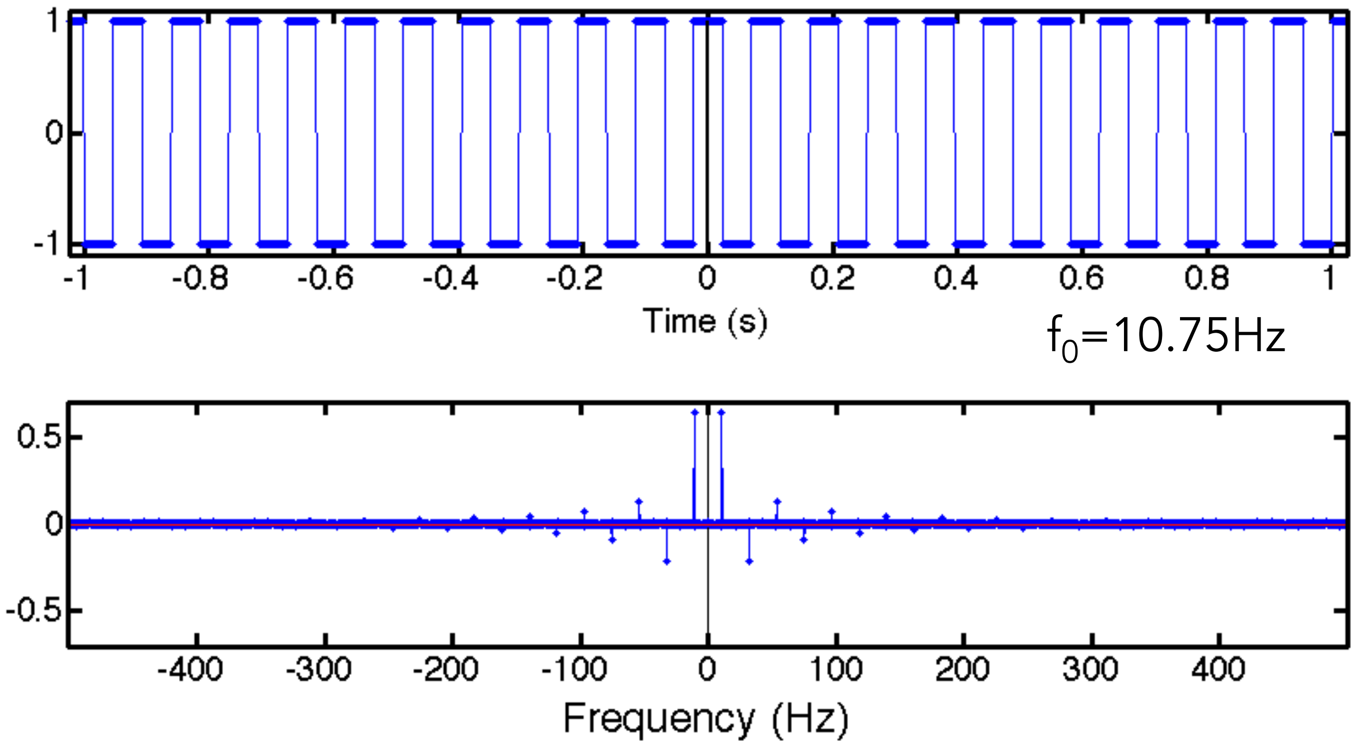
f_0=5.375Hz



Discrete Fourier transform

- Some examples – square waves

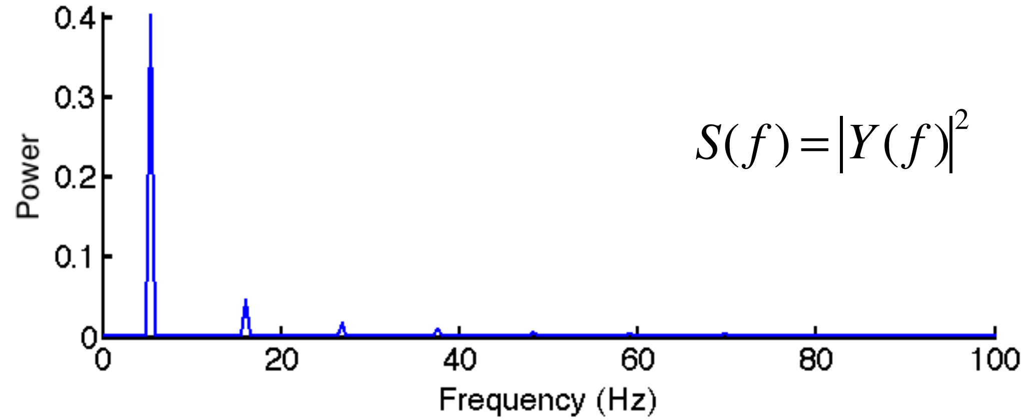
Continuous_square.m



Discrete Fourier transform

- Power spectrum– square wave

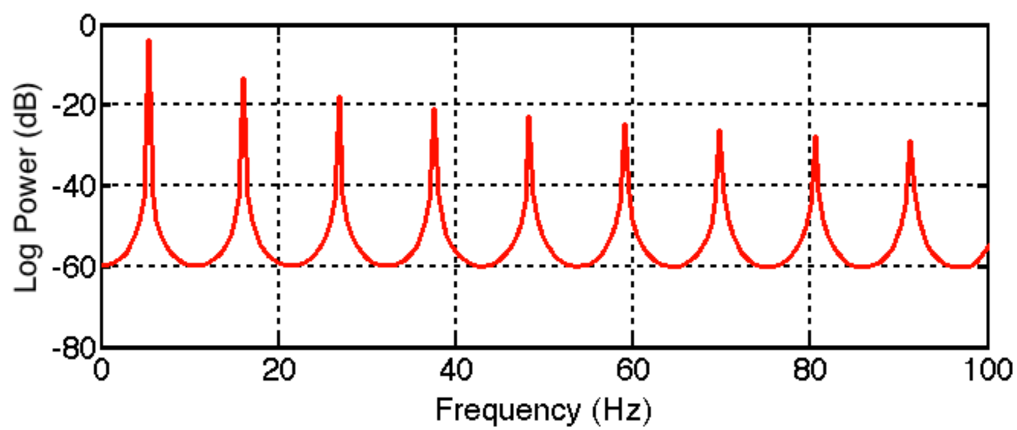
Continuous_square.m



$$S(f) = |Y(f)|^2$$

Spectrum plotted in units of decibels (dB)

$$10\log_{10} S(f)$$



Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum