Problem Set 5 (due Mar. 10)
Nonlinear network theory again

February 25, 2005

1. Group winner-take-all. Consider the network

$$\dot{x}_{ia} + x_{ia} = \left[b_{ia} + \alpha_0 x_{ia} + \alpha_1 \sum_j x_{ja} - \beta \sum_{jb} x_{jb} \right]^+$$

   We can think of the neural activities x_{ia} as arranged in a matrix. Then there is excitation with strength α_1 along each column. Also there is global inhibition with strength β, and self-excitation with strength α_0.

   (a) Find a necessary and sufficient condition on the parameters α_0, α_1, and β such that any set containing neurons in different columns is forbidden. You should use the fact that any superset of a forbidden set is forbidden.

   (b) Find a necessary and sufficient condition on the parameters α_0, α_1, and β such that any set consisting of neurons in a single column is permitted. You should use the fact that any subset of a permitted set is permitted.

   (c) Simulate the dynamics for the case where the indices i and a run from 1 to 5. Find a set of parameters that satisfies the two conditions above, and for which the neural activities do not run away to infinity. You should see that a single column of neurons always wins the competition, suppressing all other columns.

2. Stereopsis. Consider the network

$$\dot{x}_{i,j} + x_{i,j} = \left[b_{i,j} + \alpha_0 x_{i,j} + \alpha_1 x_{i+1,j+1} + \alpha_1 x_{i-1,j-1} - \beta \sum_k x_{i,k} - \beta \sum_k x_{k,j} \right]^+$$

   which is similar to the original Marr-Poggio cooperative network, except that it uses threshold linear neurons instead of binary neurons. If the neural activities $x_{i,j}$ are regarded as constituting a matrix, then there is inhibition with strength β along each row and column. Furthermore, there is nearest neighbor excitation with strength α_1 along each diagonal, and self-excitation with strength α_0.

   (a) Find the condition on the parameters α_0, α_1, and β such that any set containing two neurons in the same row or column is forbidden. This implements Marr's uniqueness constraint.

   (b) Simulate the stereopsis network with i and j running from 1 to 30. Generate a random ± 1 1d image of size 30, and use it for s_j^{right}. Let s_i^{left} be the same image, except that pixels 11 through 21 are shifted to the left by one pixel. Form the feedforward input

$$b_{i,j} = (s_i^{left} s_j^{right} + 1)/2$$

   by multiplying the left and right images, to obtain a vector that is one for a match and zero otherwise. Find a set of parameters that satisfies the condition derived above for the uniqueness constraint, and for which the neural activities do not run away to infinity. You should see a proper depth map emerge, with zero disparity on the edges, and nonzero disparity in the middle.

3. Nearest neighbor excitation. Consider the network

$$\dot{x}_i + x_i = \left[b_i + \alpha_0 x_i + \alpha_1 x_{i+1} + \alpha_1 x_{i-1} - \beta \sum_j x_j \right]^+ \tag{1}$$

with nearest neighbor and self-excitation, as well as global inhibition.

Whether an active set consisting of k adjacent neurons is permitted depends on the largest eigenvalue of the corresponding $k \times k$ submatrix $\tilde{W}_k$ of the weight matrix W,

$$\tilde{W}_k = \begin{pmatrix} \alpha_0-\beta & \alpha_1-\beta & -\beta & \dots & \dots & -\beta \\ \alpha_1-\beta & \alpha_0-\beta & \alpha_1-\beta & \dots & \dots & -\beta \\ -\beta & \alpha_1-\beta & \alpha_0-\beta & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \alpha_0-\beta & \alpha_1-\beta \\ -\beta & -\beta & \dots & \dots & \alpha_1-\beta & \alpha_0-\beta \end{pmatrix} \tag{2}$$

This is a Toeplitz matrix, and cannot be diagonalized so easily. Instead we will construct a closely related circulant matrix, which can be diagonalized by the Fourier modes. Then we'll argue that the largest eigenvalue of the circulant matrix is the same as the largest eigenvalue of $\tilde{W}_k$.

(a) Add a row and column to $\tilde{W}_k$ to make it a $(k+1)\times(k+1)$ circulant matrix (you encountered a circulant matrix in Assignment 5). Find the eigenvectors and eigenvalues of the circulant matrix.

(b) In the following you may make use of the the interlacing theorem. Suppose that a row and column are added to a symmetric $k \times k$ matrix A to generate a symmetric $(k+1)\times(k+1)$ matrix $\hat{A}$. Then the eigenvalues of $\hat{A}$ interlace the eigenvalues of A. More precisely, let the eigenvalues of A be arranged in increasing order, $\lambda_1 \le \dots \le \lambda_k$. Similarly arrange the eigenvalues of $\hat{A}$ as $\hat{\lambda}_1 \le \dots \le \hat{\lambda}_{k+1}$. Then

$$\hat{\lambda}_1 \le \lambda_1 \le \hat{\lambda}_2 \le \lambda_2 \le \dots \le \lambda_k \le \hat{\lambda}_{k+1}$$

Now use this theorem to prove that the largest eigenvalue of $\tilde{W}_k$ for $k > 1$ is

$$\lambda_{max}(\tilde{W}_k) = \alpha_0 + 2\alpha_1 \cos\frac{2\pi}{k+1}$$

(c) This has an important implication for permitted and forbidden sets. Suppose that α_0 and α_1 are such that $\theta \in (0,\pi)$ can be defined by

$$\cos\theta = \frac{1-\alpha_0}{2\alpha_1}$$

Show that the inequality

$$\frac{2\pi}{k_{max}+1} > \theta > \frac{2\pi}{k_{max}+2}$$

with $k_{max} > 1$ implies that all active sets of k_{max} or less adjacent neurons are permitted, while all active sets of more than k_{max} adjacent neurons are forbidden.

(d) Simulate the dynamics (1) with uniform input $b_i = 1$. Using the results derived above, choose three sets of parameters so that k_{max} is 5, 10, and 15. Show that your simulations converge to steady states with these widths.

Extra credit: Can you find *all* eigenvalues and eigenvectors of $\tilde{W}_k$? (It's easy to find half of them.)