An airfoil with chord $c$ is moving at velocity $U$ with zero angle of incidence through the air, as shown in the figure below:

The air is not motionless, but rather has variations in the vertical velocity, $w$. As the airfoil flies through this gust field, the leading edge of the airfoil “sees” a variation in the angle of attack. If $w$ is small compared to $U$, then the angle of attack change seen by the airfoil is $\alpha = w/U$. Since the velocity profile varies in space, the angle of attack seen by the airfoil is a function of time, $\alpha(t)$.

One might expect that the lift coefficient of the airfoil is just

$$C_L(t) = 2\pi \alpha(t)$$

However, the airfoil does not respond instantaneously as the airfoil encounters the gust. If the airfoil encounters a “sharp-edged gust,” so that the apparent change in the angle of attack is a step function in time,

$$\alpha(t) = \alpha_0 \sigma(t)$$

then the change in lift is given by

$$C_L(t) = 2\pi \alpha_0 \psi(\tilde{t})$$

where $\tilde{t} = 2Ut/c$ is the dimensionless time. $\psi(\tilde{t})$ is the Küssner function, and is the step response of the airfoil (neglecting multiplicative constants), if the input is considered to be the vertical gust at the leading edge as a function of time, and the output is considered to be the lift as a function of time. The Küssner function can be approximated as

$$\psi(\tilde{t}) = \begin{cases} 0, & \tilde{t} < 0 \\ 1 - \frac{1}{2} e^{-0.13\tilde{t}} - \frac{1}{2} e^{-\tilde{t}}, & \tilde{t} \geq 0 \end{cases}$$

Assuming that the airfoil acts as an LTI system, determine and plot the lift coefficient, $C_L(t)$, and the gust velocity, $w(t)$, for the following conditions:

$$c = 1 \text{ m}$$

$$U = 1 \text{ m/s}$$

$$w(t) = \begin{cases} 0 \text{ m/s}, & t < 0 \text{ s} \\ 0.1 \cdot (1 - e^{-2t}) \text{ m/s}, & t \geq 0 \text{ s} \end{cases}$$