Elementary Logic

• Proposition is sentence that can be either true or false, not both

• Symbolic notations for manipulating logic of propositions
  - $\neg$ “not” or negation
  - $\land$ “and”
  - $\lor$ “or”
  - $\leftrightarrow$ “if and only if”
  - $\rightarrow$ “implies”

• Quantifiers
  - $\forall x p(x)$ “is true if for all $x$ in $U$, $p(x)$ is true”
  - $\exists x p(x)$ “is true if there exists an $x$ such that $p(x)$ is true”
Elementary Logic

• The proposition \( q \rightarrow p \) is called the converse of \( p \rightarrow q \), and \( \neg q \rightarrow \neg p \) is the contrapositive of \( p \rightarrow q \)

• **Example:**
  - Give the converse of the following propositions
    • \( q \rightarrow r \)
    • If I am smart, then I am rich
    • If \( x^2 = x \), then \( x = 0 \) or \( x = 1 \)
    • If \( 2 + 2 = 4 \), then \( 2 + 4 = 8 \)
  - Give the contrapositives for the propositions above

• Breaking assertions into component propositions
  - look for the logical operators!

**Example:**
If I go to Harry’s or go to the country I will not go shopping.

P: I go to Harry’s
Q: I go to the country
R: I will go shopping
If......P......or.....Q.....then....not.....R

\[(P \lor Q) \rightarrow \neg R\]
Elementary Logic

- Let p, q, r be the following propositions
  - p = “it is raining”
  - q = “the sun is shining”
  - r = “there are clouds in the sky”
- Translate the following into logical notation, using p, q, r and logical connectives
  - It is raining and the sun is shining
  - If it is raining, then there are clouds in the sky
  - If it is not raining, then the sun is not shining and there are clouds in the sky
  - The sun is shining if and only if it is not raining
  - If there are no clouds in the sky, then the sun is shining

Convert the following into predicate logic sentences

- Shamu can do every trick
- Shamu can do any trick
- Shamu cannot do every trick
- If any whale can do a trick, Shamu can
- If every whale can do a trick, Shamu can
- If any whale can do a trick, any whale can do a trick
\[ T_\alpha = \alpha \text{ is a trick} \]
\[ W_\alpha = \alpha \text{ is a whale} \]
\[ S_\alpha = \text{Shamu can do } \alpha \]
\[ C_\alpha = \alpha \text{ can do a trick} \]
\[ s = \text{Shamu} \]

\[ \forall x (T_x \rightarrow S_x) \] [Shamu can do every trick]
\[ \neg \forall x (T_x \rightarrow S_x) \] [Shamu cannot do every trick]

[If any whale can do a trick, Shamu can]
\[ \forall x (W_x \text{ and } C_x \rightarrow C_s) \]

[If every whale can do a trick, Shamu can]
\[ \forall x (W_x \rightarrow C_x) \rightarrow C_s \]
Proof by Cases

- Consider several cases that are exhaustive—i.e., that include all the possibilities
- **Example**: Prove that $n^2 - 2$ is not dividable by 5 for any positive integer
  - Case 1: $n=5k$
  - Case 2: $n=5k+1$
  - Case 3: $n=5k+2$
  - Case 4: $n=5k+3$
  - Case 5: $n=5k+4$

Proof by Contradiction

- We show that a conclusion holds by assuming it does not. If this leads to ‘nonsense’ contrary to reality (or hypotheses), then we have reached a contradiction.
- **Example**: Prove that there are infinitely many primes.
- **Example**: Prove that the sum of a rational number and an irrational number is always irrational.
Direct Proof

- Show that a given statement is true by simple combination of existing theorems with/without some mathematical manipulations
  \(- H_1 \land H_2 \land \ldots \land H_n \Rightarrow C\)

- Proof of the contrapositive (indirect proof)
  \(- \neg C \Rightarrow \neg (H_1 \land H_2 \land \ldots \land H_n)\)

Direct or Indirect Proof?

- **Example**: Let \(m, n \in \mathbb{N}\). Prove that if \(m+n \geq 73\) then \(m \geq 37\) or \(n \geq 37\)
Proof by Contradiction

- \((H_1 \land H_2 \land ... \land H_n) \land \neg C \Rightarrow a \text{ contradiction}\)

- **Example**: If \(5n+6\) is odd, then \(n\) is odd

- **Example**: prove that at least 4 of any 22 days must fall on the same day of the week

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- **Example:** Convert each of the following arguments into logical notation using the suggested variables. Then provide a formal proof.
  - “if my computations are correct and I pay the electric bill, then I will run out of money. If I do not pay the electric bill, the power will be turned off. Therefore, if I do not run out of money and the power is still on, then my computations are incorrect”.
    
    \[(c, b, r, p)\]

- Let
  - \(c := “my computations are correct”\)
  - \(b := “I pay the electric bill”\)
  - \(r := “I run out of money”\)
  - \(p := “the power stays on”\)

- Then theorem is:
  - if \((c \land b) \implies r\) and \(\neg b \implies \neg p\), then \((\neg r \land p) \implies \neg c\)
Mathematical Induction

- Let \( p(m), p(m+1), \ldots, p(n) \) be a sequence of propositions. If
  \( p(m) \) is true, and
  \( p(k+1) \) is true whenever \( p(k) \) is true and \( m \leq k < n \),
  then all propositions are true.

- **Example:** for each positive integer \( n \), let \( p(n) \) be “\( n! > 2^n \),” a proposition that we claim is true for \( n \geq 4 \).
  To give a proof by induction, we verify that \( p(n) \) for \( n=4 \), and then show
  \( p(k+1) \) is true whenever \( 4 \leq k \) and \( k! > 2^k \), then \( (k+1)! > 2^{k+1} \).