Fluids – Lecture 14 Notes

1. Helmholtz Equation
2. Incompressible Irrotational Flows

Reading: Anderson 3.7

Helmholtz Equation

Derivation (2-D)

If we neglect viscous forces, the x- and y-components of the 2-D momentum equation can be written as follows.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + g_x \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial y} + g_y \tag{2}$$

We now take the curl of this momentum equation by performing the following operation.

$$\frac{\partial}{\partial x}\left\{y\text{-momentum (2)}\right\} - \frac{\partial}{\partial y}\left\{x\text{-momentum (1)}\right\}$$

If we assume that ρ is constant (low speed flow), the two pressure derivative terms cancel. Since the gravity components g_x and g_y are generally constant, these also disappear when the curl's derivatives are applied. Using the product rule on the lefthand side, the resulting equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0$$

We note that the quantity inside the parentheses is merely the z-component of the vorticity $\xi \equiv \partial v/\partial x - \partial u/\partial y$, so the above equation can be more compactly written as

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + \xi\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0$$

We further note that the quantity in the brackets is the divergence of the velocity, which in low speed flow must be zero because of mass conservation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \equiv \nabla \cdot \vec{V} = 0 \qquad \text{(mass conservation equation)}$$

This gives the following final result.

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = 0$$

$$\text{or}\ldots \qquad \frac{D\xi}{Dt} = 0 \tag{3}$$

Equation (3) is the 2-D form of the *Helmholtz Equation*, which governs the vorticity field $\xi(x, y, t)$ in inviscid flow.

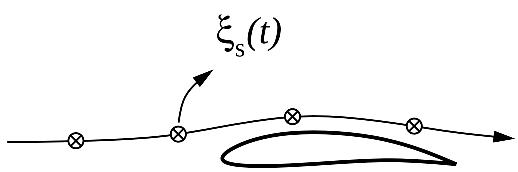
Interpretation and Implications

Consider a microscopic sensor drifting with the flow (along a pathline) near an airfoil. The sensor's time-trace signal $\xi_s(t)$ is the vorticity at the sensor's instantaneous location. Equation (3) implies that this vorticity signal $\xi_s(t)$ will have zero time rate of change, since we know that

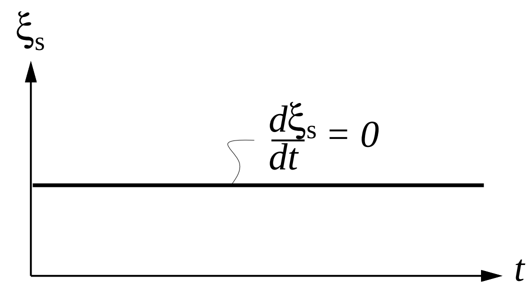
$$\frac{d\xi_s}{dt} = \frac{D\xi}{Dt} = 0$$

Hence, the vorticity along a pathline must be constant.

$$\xi_s = \text{constant}$$



Vorticity sensor moving along streamline



Constant sensor output

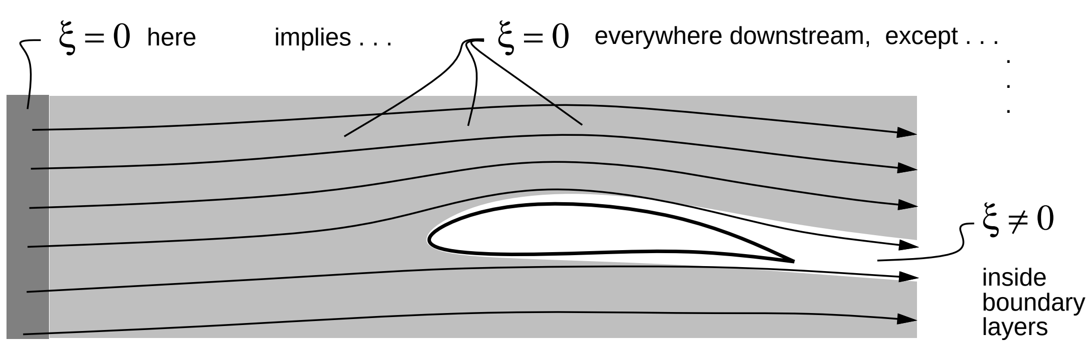
Furthermore, this constant value must be determined far upstream of the airfoil where the streamline originates. If the freestream flow velocity $\vec{V}_\infty$ is uniform (either zero or some constant), then

$$\xi_s = \nabla \times \vec{V}_\infty = 0$$

This is true for <u>all</u> streamlines which originate in the uniform upstream flow, so that the entire flowfield must be irrotational, as shown in the figure.

$$\xi(x, y, t) \equiv \nabla \times \vec{V} = 0 \qquad \text{(if upstream flow is uniform)}$$

The one exception is that $\xi \neq 0$ for any streamline which is affected by viscous forces. For these streamlines the Helmholtz equation (3) does not hold, since here the viscous forces are not negligible, as was assumed at the outset.



For 3-D flows, it is possible to derive a more general 3-D Helmholtz equation. From this we can also conclude that 3-D flows which are initially uniform are irrotational downstream. These 3-D derivations are considerably more cumbersome, and will not be attempted here.

Incompressible, Irrotational Flows

Governing Equations

The mass conservation equation for an incompressible flow states that the velocity field has zero divergence.

$$\nabla \cdot \vec{V} \;=\; 0 \tag{4}$$

The Helmholtz equation implies that an inviscid flow which is uniform upstream must be irrotational, and can therefore be expressed in terms of a potential function.

$$\vec{V} \;=\; \nabla \phi$$

Substituting this into the divergence equation (4) gives

$$\nabla \cdot (\nabla \phi) \;=\; \nabla^2 \phi \;=\; 0 \tag{5}$$

This is *Laplace's Equation*. In Cartesian coordinates, with $\phi = \phi(x, y, z)$, Laplace's equation is explicitly given by

$$\nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \;=\; 0$$

In cylindrical coordinates, with $\phi = \phi(r, \theta, z)$, it has the form

$$\nabla^2 \phi \;=\; \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) \;+\; \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \;=\; 0$$

For 2-D problems, the stream function can be employed in lieu of the potential function. The requirement that the flow be irrotational leads to

$$\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \;=\; -\frac{\partial}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} \;&=\; 0 \\ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \;&=\; 0 \end{aligned} \tag{6}$$

Hence, if the stream function is employed, it must also satisfy Laplace's equation.

Superposition

Laplace's equation is *linear*. If $\phi_1(x, y, z)$, $\phi_2(x, y, z)$ are valid solutions, then their sum

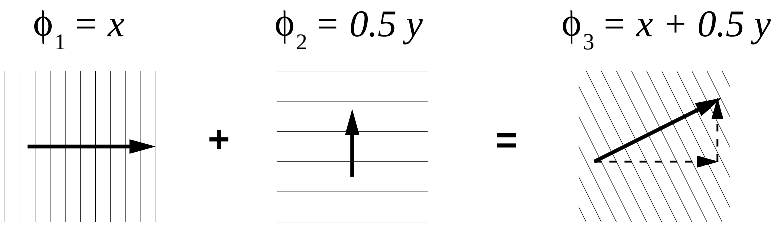
$$\phi_3(x, y, z) \;=\; \phi_1 \;+\; \phi_2$$

is another valid solution. The corresponding velocities can therefore be obtained via vector summation.

$$\vec{V}_3(x, y, z) \;=\; \nabla \phi_3 \;=\; \nabla \left(\phi_1 + \phi_2 \right) \;=\; \vec{V}_1 \;+\; \vec{V}_2$$

This is the principle of *superposition*, which allows constructing complex flowfields from any number of relatively simple components. The figure shows an example of two uniform flows being superimposed into a third uniform flow. Stream functions can be superimposed in the same manner. The pressure field in each case is obtained using Bernoulli's equation.

$$p_1(x, y, z) \;=\; p_o - \frac{1}{2}\rho \left| \nabla \phi_1 \right|^2 \quad , \quad p_2(x, y, z) \;=\; p_o - \frac{1}{2}\rho \left| \nabla \phi_2 \right|^2 \quad \ldots \;\; \text{etc}$$



Boundary Conditions

In order to solve Laplace's equation, it is necessary to apply *boundary conditions* at all boundaries of the flowfield. For most aerodynamic problems these fall into two categories.

Infinity Boundary Conditions

The flow far away from the body must approach the freestream velocity. Choosing the x axis to be aligned with the freestream direction, we require

$$u \;=\; \frac{\partial \phi}{\partial x} \;=\; V_\infty \qquad\qquad \text{(at infinity)}$$

If a stream function is used, the corresponding boundary condition would be

$$u \;=\; \frac{\partial \psi}{\partial y} \;=\; V_\infty \qquad\qquad \text{(at infinity)}$$

Wall Boundary Conditions

The flow adjacent to the wall is physically constrained to flow parallel, or *tangent* to the wall. If the velocity vector is tangent, then its normal component must clearly be zero.

$$\vec{V} \cdot \hat{n} \;=\; (\nabla \phi) \cdot \hat{n} \;=\; \frac{\partial \phi}{\partial n} \;=\; 0 \qquad\qquad \text{(on wall)}$$

The boundary condition on the alternative stream function is

$$\vec{V} \cdot \hat{n} \;=\; \frac{\partial \psi}{\partial s} \;=\; 0 \qquad\qquad \text{(on wall)}$$

where s is the arc length along the surface. This can also be specified as

$$\psi(s) \;=\; \text{constant} \qquad\qquad \text{(on wall)}$$