**KEY CONCEPTS FOR MATERIALS AND STRUCTURES**

Handout for Spring Term Quizzes

**Basic modeling process for 1-D structural members**
1. Idealize/model – make assumptions on geometry, load/stress and deformations
2. Apply governing equations (e.g. equations of elasticity)
3. Invoke known boundary conditions to derive constitutive relations for structure (load-deformation, load-internal stress etc.)

**Analytical process for 1-D structural members**
1. Idealize/model – assumptions on geometry, load/stress and deformations
2. Draw free body diagram
3. Apply method of sections to obtain internal force/moment resultants
4. Apply structural constitutive relations to relate force/moment resultants to
   a) internal stresses
   b) deformations (usually requires integration – invoking boundary conditions)

**Elastic bending formulae**

Based on convention for positive bending moments and shear forces:

\[ q = \frac{dS}{dx}, \quad S = \frac{dM}{dx} \]

Bending of a symmetric cross section about its neutral axis (mid plane for a cross-section with two orthogonal axes of symmetry).

\[ \sigma_{xx} = -\frac{Mz}{I}, \quad M = EI \frac{d^2w}{dx^2}, \quad \sigma_{xz} = -\frac{SQ}{lb} \]

where \( \sigma_{xx} \) is the axial (bending) stress, \( M \) is the bending moment at a particular cross-section, \( I \) is the second moment of area about the neutral axis, \( z \) is the distance from the neutral axis, \( E \) is the Young’s modulus of the material, \( w \) is the deflection, \( x \) is the axial coordinate along the beam, \( \sigma_{xz} \) is the shear stress at a distance \( z \) above the neutral axis, \( S \) is the shear force at a particular cross section, \( Q \) is the first moment of area of the cross-section from \( z \) to the outer ligament, \( b \) is the width of the beam at a height \( b \) above the neutral axis.

**Second moment of area** \[ I = \int z^2 dA \]

Standard solutions:

- Rectangular area, breadth \( b \), depth \( h \): \[ I = \frac{bh^3}{12} \]  Solid circular cross-section, radius \( R \): \[ I = \frac{\pi R^4}{4} \]

- Isosceles Triangle, depth \( h \), base \( b \): \[ I = \frac{bh^3}{36} \] (note centroid is at \( h/3 \) above the base)

**Parallel axis theorem:**
If the second moment of area of a section, area A, about an axis is $I$ then the second moment of area $I'$ about a parallel axis, a perpendicular distance $d$ away from the original axis is given by:

$$I' = I + Ad^2$$

**First moment of area**
The first moment of area of a section between a height $z$ from the neutral plane and the top surface (outer ligament) of the section is given by:

$$Q = \frac{h}{2} \int z dA$$

### Standard solutions for deflections of beams under commonly encountered loading

<table>
<thead>
<tr>
<th>Configuration</th>
<th>End slope, dw/dx (x=L)</th>
<th>End deflection, w(L)</th>
<th>Central deflection, w(L/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Beam with distributed load" /></td>
<td>$\frac{ML}{EI}$</td>
<td>$\frac{ML^2}{2EI}$</td>
<td></td>
</tr>
<tr>
<td><img src="image2" alt="Beam with point load" /></td>
<td>$\frac{PL^2}{2EI}$</td>
<td>$\frac{PL^3}{3EI}$</td>
<td></td>
</tr>
<tr>
<td><img src="image3" alt="Beam with distributed load" /></td>
<td>$\frac{q_0 L^3}{6EI}$</td>
<td>$\frac{q_0 L^4}{8EI}$</td>
<td></td>
</tr>
<tr>
<td><img src="image4" alt="Beam with point load" /></td>
<td>$\frac{PL^2}{16EI}$</td>
<td>$\frac{PL^3}{48EI}$</td>
<td></td>
</tr>
<tr>
<td><img src="image5" alt="Beam with distributed load" /></td>
<td>$\frac{q_0 L^3}{24EI}$</td>
<td></td>
<td>$\frac{5q_0 L^4}{384EI}$</td>
</tr>
</tbody>
</table>
Singularity functions

Integration of singularity functions: 
\[ \int_{-\infty}^{x} (x-a)^n \, dx = \frac{(x-a)^{n+1}}{n+1}, \quad n \geq 0 \]

\[ \int_{-\infty}^{x} (x-a)_-^2 \, dx = (x-a)_{-1} \]

\[ \int_{-\infty}^{x} (x-a)_-^1 \, dx = (x-a)^0 \]

Torsion of round shafts

An internal torque resultant, T, generates a circumferential shear stress, \( \tau \), at a radius \( r \), and twist per unit length, \( \frac{d\phi}{dx} \), where:

\[ \tau = \frac{Tr}{J} \]

\[ T = GJ \frac{d\phi}{dx} \]

G is the shear modulus of the material and J is the second polar moment of area given by:

\[ J = \int_{A} r^2 \, dA \]

For a solid circular cross section, radius R:

\[ J = \frac{\pi R^4}{2} \]

For a thin walled circular tube, radius R, thickness t:

\[ J = 2\pi R^3 t \]
Elastic buckling of columns

The general governing equation for the transverse (buckling), w, of a uniform column of bending stiffness EI, under an axial load P is: \( \frac{d^2w}{dx^2} + \frac{P}{EI} x = M_0 \). Where \( M_0 \) is a constant.

General solutions are of the form:
\[
w = A\sin\left(\frac{P}{EI} x\right) + B\cos\left(\frac{P}{EI} x\right) + Cx + D.
\]

In general the elastic critical load, \( P_{cr} = cP_E \), where the factor \( c \) depends on the boundary conditions and the order of the buckling mode, and \( P_E \) is the Euler Load for a perfect, pin ended column of length, \( L \) buckling into a half sine wave given by:
\[
P_E = \frac{\pi^2 EI}{L^2}.
\]

Yield and Plasticity of Metals

Uniaxial loading of a bar, initial length \( \ell_0 \), cross-sectional area \( A_0 \) past yield point: Define nominal, true stress and nominal and true strain:
\[
\sigma_n = \frac{P}{A_0}, \quad \sigma_t = \frac{P}{A}, \quad \varepsilon_n = \frac{\Delta \ell}{\ell_0} = \frac{\ell - \ell_0}{\ell_0}, \quad \varepsilon_t = \int \frac{d\ell}{\ell_0} = \ln \left( \frac{\ell}{\ell_0} \right)
\]

Since volume is conserved: \( A_0 \ell_0 = A \ell \) obtain: \( \sigma_t = \sigma_n \left( 1 + \varepsilon_n \right) \) and \( \varepsilon_t = \ln \left( 1 + \varepsilon_n \right) \)

Work of deformation per unit volume: \( U = \int_{\varepsilon_{n1}}^{\varepsilon_{n2}} \sigma_n d\varepsilon_n = \int_{\varepsilon_{t1}}^{\varepsilon_{t2}} \sigma_t d\varepsilon_t \)

Elastic Strain Energy (for linear elastic deformation): \( U = \frac{\sigma_n^2}{2E} \)

For multiaxial stress states models for yield:

Tresca: \( \max \left( |\sigma_I - \sigma_{II}|, |\sigma_{II} - \sigma_{III}|, |\sigma_{III} - \sigma_I| \right) \geq \sigma_y \)

Von Mises: \( \left( \sigma_I - \sigma_{II} \right)^2 + \left( \sigma_{II} - \sigma_{III} \right)^2 + \left( \sigma_{III} - \sigma_I \right)^2 \geq 2\sigma_y^2 \)

Where \( \sigma_I \) etc are the principal stresses and \( \sigma_y \) is the uniaxial yield strength

Hardness \( H = \frac{F_{\text{indentation}}}{A_{\text{indentation}}} = 3\sigma_y \)

In a uniaxial tension test, necking occurs when: \( \frac{d\sigma_t}{d\varepsilon_t} = \sigma_t \)
Transformation of Stress and Strain via Mohr’s Circle:

Mohr's circle is a geometric representation of the 2-D transformation of stresses.

**Construction:** Given the state of stress shown below for an infinitesimal element, with the following definition (by Mohr) of positive and negative shear:

"Positive shear would cause a clockwise rotation of the element about the element center."

Thus: \( \sigma_{21} \) (below) is plotted positive \( \sigma_{12} \) (below) is plotted negative:

Principal stresses correspond to points G, F. Max shear at H, H’.

Note that angles are doubled on the Mohr’s circle relative to the physical problem.

Note that a Mohr’s circle can only be drawn stresses in a plane perpendicular to a principal direction.
**Strengthening Mechanisms**

Precipitate Strengthening: \[ \Delta \tau_y = \frac{Gb}{L} \] where G = shear modulus, b = Burgers vector, L = particle spacing

Solid Solution strengthening: \[ \Delta \tau_y \propto \sqrt{c} \] where c = concentration of alloying elements

Work Hardening: \[ \Delta \tau_y \propto \gamma^m \] where \( \gamma \) = shear strain, \( m \) = exponent (0.01-0.5)

Grain Boundary Effect: \[ \Delta \tau_y \propto \frac{1}{\sqrt{d}} \] where d = grain size

**Fracture and Fatigue**

Fast fracture occurs when: \[ dW \geq dU_{el} + G_c dA \]
where \( W \) = external work, \( U_{el} \) = elastic strain energy, \( G_c \) is the material’s toughness and A is the area of crack surface.

Can also be written: \( K \geq K_c \) Where \( K_c \) is the fracture toughness and K is the stress intensity factor given by:
\[ K = Y\sigma \sqrt{\pi a} \]
where \( Y \) is a factor which depends on the crack and component shape (≈1), a is the crack length and \( \sigma \) the applied stress

For many metals fatigue crack growth is of the form:
\[ \frac{da}{dN} = A\Delta K^n \]
where A and n are empirically determined constants.