LINEAR, TIME-ININVARIANT SYSTEMS

Any system with input $u(t)$, output $y(t)$
can be represented as

$$\begin{array}{c}
u(t) \\
\downarrow \\
G \\
\uparrow \\
y(t)
\end{array}$$

The output $y(t)$ is a functional of the
input signal $u(t)$:

$$y(t) = G[u(t)]$$

Note: We should really write that

$$y(\cdot) = \mathcal{G}[u(\cdot)]$$

meaning that $y(t)$ at each $t$ depends
on $u(t)$ at all values of $t$.

We will be most interested in $\mathcal{G}[\cdot]$
which is linear, time-invariant.
Linearity

A system is linear if
\[G[a u_1(t) + b u_2(t)] = a G[u_1(t)] + b G[u_2(t)]\]
for all \(a, b, u_1(t), u_2(t)\)

System is linear \(\iff\) superposition always holds

Linear system:

Nonlinear systems: real circuits, airplane, helicopter, spacecraft.

Almost linear system: real circuit, airplane, helicopter, spacecraft.

Message: Although all physical systems are nonlinear, many can be modeled as linear for some purposes.

Since linear systems are much simpler than nonlinear systems, do this whenever possible.
Time Invariance

A system is time-invariant if

\[ y(t) = G[u(t)] \Rightarrow y(t-T) = G[u(t-T)] \]

for all \( T, u(t) \).

In words, shifting the input in time shifts the output in time the same amount.

Example - An aircraft is nonlinear, because lift is a nonlinear function of speed \( \sim v^2 \) and attitude (stall).

An aircraft is time-varying, because configuration, weight, etc., change over time.

In Unified, will consider only linear, time-invariant (LTI) systems, because we know so much about them.
The step response
The unit step is defined by

\[ \sigma(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \]

The step response of an LTI system is the output of the system when the input to the system is a unit step.

Example What is step response of

![Circuit Diagram]

\[ u(t) = \sigma(t) \]

\[ y(t) = g_s(t) \]

\[ g_s(t) \]
Solution Use node method:

At e₁, \( \frac{e₁ - u}{R} + c \frac{d}{dt} (e₁ - 0) = 0 \)

\[ \Rightarrow \frac{d}{dt} e₁(t) + \frac{1}{RC} e₁(t) = \frac{1}{RC} u(t) \]

For \( t \geq 0, \ u(t) = 1. \) So solve

\[ \frac{d}{dt} e₁(t) + \frac{1}{RC} e₁(t) = \frac{1}{RC} \]

subject to initial condition

\[ e₁(0) = 0 \]

Express solution as

\[ e₁(t) = eₚ(t) + eₕ(t) \]

\[ \uparrow \]

\[ \uparrow \text{homogeneous} \]

\[ \text{particular} \]

Find particular, then homogeneous.
Particular Solution -
Since input is constant, assume

\[ e_p(t) = E = \text{constant} \]

Plug into equation:

\[ \Delta^2 \frac{E}{RC} + \frac{1}{RC} E = \frac{1}{RC} \]

\[ 0 \]

\[ \Rightarrow E = 1 \quad \Rightarrow \quad e_p(t) = 1 \]

Homogeneous Solution -
Assume solution is of the form

\[ e_h(t) = E e^{st} \]

Plug into homogeneous equation:

\[ \frac{d}{dt} \left( E e^{st} \right) + \frac{1}{RC} \left( E e^{st} \right) = 0 \]

\[ \Rightarrow E s e^{st} + \frac{1}{RC} E e^{st} = 0 \]

\[ \Rightarrow s + \frac{1}{RC} = 0 \quad \Rightarrow \quad s = -1/RC \]

\[ \Rightarrow e_h(t) = E e^{-t/RC} \]
Total Solution:

\[ e_i(t) = 1 + E e^{-t/RC} \]

Initial condition is \( e_i(0) = 0 \)

\[ \Rightarrow 1 + E e^{-0/RC} = 1 + E = 0 \]

\[ \Rightarrow E = -1 \]

So,

\[ g_s(t) = y(t) = e_i(t) = \begin{cases} 
1 - e^{-t}, & t \geq 0 \\
0, & t < 0 
\end{cases} \]

Note—could have found particular solution using impedance methods. Later, will find total solution using impedance methods.
Amazing fact:

The step response of an LTI system completely characterizes the system!

That is, if you know the step response, you can find the response to any input.

\[ u(t) \]

Can represent \( u(t) \) arbitrarily well by summing a series of scaled, delayed steps.

\[ \Rightarrow \text{response is sum of scaled, delayed step responses, by superposition.} \]

Next time, will do the summation, to find \( y(t) \) in terms of \( g_s(t) \), \( u(t) \).