The Superposition Integral

Another way to look at the problem:

\( u(t) \) is (approximately) a sum of pulses

\[
  u(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \left[ \sigma(t - \tau_n) - \sigma(t - \tau_{n+1}) \right]
\]

Then, by superposition,

\[
  y(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \left[ g_s(t - \tau_n) - g_s(t - \tau_{n+1}) \right]
\]
\[ y(t) \approx \sum_{n=-\infty}^{\infty} u(\tau_n) \left[ \frac{g_s(t-\tau_n) - g_s(t-\tau_{n+1})}{\tau_{n+1} - \tau_n} \right] \left( \tau_{n+1} - \tau_n \right) \]

In the limit,

\[ y(t) = \int_{-\infty}^{\infty} g(t-\tau) u(\tau) \, d\tau \]

This is the "Superposition Integral." 

\( g(t) \) is

\[ g(t) = \frac{d}{dt} g_s(t) \]

= "Impulse Response"

Why do we call it that?
The Impulse Function

A "pulse" is a short signal, often a square wave. The unit pulse function is

\[ \delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 \leq t < \varepsilon \\ 0, & \text{else} \end{cases} \]

\[ = \frac{1}{\varepsilon} \left[ \sigma(t) - \sigma(t-\varepsilon) \right] \]

In the limit as \( \varepsilon \to 0 \), we get an "impulse":

\[ \delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) \]

\[ = \lim_{\varepsilon \to 0} \frac{\sigma(t) - \sigma(t-\varepsilon)}{\varepsilon} \]

\[ = \frac{d}{dt} \sigma(t) \quad \text{unit area, zero width} \]
We have to be careful:

1) $\delta(t)$ is not really a function—

\[
\delta(t) = \begin{cases} 
\infty & \text{if } t=0 \\
0 & \text{if } t \neq 0 
\end{cases}
\]

is not enough to define an impulse.

2) We could have started with a pulse of any shape, and we would get the same result.

3) An impulse is really defined by what it does:

\[
\int_{-\infty}^{t} \delta(\tau) \, d\tau = \delta(t)
\]

What is the response to an impulse?
The impulse response

\[ \delta(t) \xrightarrow{G} g(t) = ? \]

\[ \delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ r(t) - r(t + \varepsilon) \right] \]

\[ \Rightarrow g(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ g(s)(t) - g(s)(t + \varepsilon) \right] \]

\[ = \frac{d}{dt} g_s(t) \]

In words:

The impulse response is the derivative of the step response, because the unit impulse is the derivative of the unit step.
An Alternative Derivation of the Superposition Integral

Recall Duhamel's Integral:

\[ y(t) = g_s(t) \ u(0) + \int_0^\infty g_s(t-\tau) \ u'(\tau) \, d\tau \]

This form is awkward since we have to use derivative of \( u(t) \). So integrate by parts:

\[ \int u \, dv = uv - \int v \, du \]

\[ u = g_s(t-\tau) \quad \Rightarrow \quad dv = -g_s'(t-\tau) d\tau \]

\[ dv = u'(\tau) d\tau \quad \Rightarrow \quad v = u(\tau) \]

Therefore,

\[ y(t) = g_s(t) \ u(0) + \left. g_s(t-\tau) u(\tau) \right|_{\tau=0}^\infty \]

\[ + \int_0^\infty g_s'(t-\tau) u(\tau) \, d\tau \]

Recall \( g(t) \equiv g_s'(t) \). Then
\[ y(t) = g_s(t) u(0) + g_s(-\infty) u(\infty) \]
\[ \quad - g_s(t) u(0) \]
\[ \quad + \int_{0}^{\infty} g(t-\tau) u(\tau) d\tau \]

Because \( g_s(-\infty) = 0 \)

\[ \Rightarrow y(t) = \int_{0}^{\infty} g(t-\tau) u(\tau) d\tau \]

More generally, \( u(\tau) \) need not be zero for \( t < 0 \). So,

\[ y(t) = \int_{-\infty}^{\infty} g(t-\tau) u(\tau) d\tau \]