Useful Properties of Unit Impulse

Have already derived that

\[ \delta(t) = \frac{d}{dt} \sigma(t) \]

\[ \Rightarrow \sigma(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \]

Recall that \( \delta(t) = 0, \ t \neq 0 \). Therefore,

\[ f(t) \delta(t) = f(a) \delta(t) \]

if \( f(t) \) "well-behaved" near \( t=0 \).

Example: \( f(t) = \cos t \)

\[ \delta(t) \cos t = \delta(t) \cos(0) = \delta(t) \]
Example: \( \tau(t) \delta(t) = ? \)

\[
\begin{array}{c}
\text{\( \tau(t) \)} \\
\text{\( \times \)} \\
\text{\( \delta(t) \)} \\
\end{array}
\]

= undefined

because \( \delta(t) \) is discontinuous at \( t=0 \)

Let's find \( f(t) \times \delta(t) \), \( \delta(t) \times f(t) \):

\[
f(t) \times \delta(t) = \int_{-\infty}^{\infty} f(t-\tau) \delta(\tau) \, d\tau
\]

= \int_{-\infty}^{\infty} f(t) \delta(\tau) \, d\tau

\text{\( \uparrow \) value of \( f(t-\tau) \) @ \( \tau=0 \)}

= \int_{-\infty}^{\infty} \delta(\tau) \, d\tau = f(t)

\text{area = 1}

\[
\delta(t) \times f(t) = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau) \, d\tau
\]

\text{\( \uparrow \) impulse at \( \tau = t \)}

= \int_{-\infty}^{\infty} \delta(t-\tau) f(t) \, d\tau

\text{\( \uparrow \) value at \( \tau = t \)}
\[ f(t) = \int_{-\infty}^{\infty} \delta(t-\tau) \, d\tau = f(t) \]

So,

\[ f(t) * \delta(t) = f(t) \]

"The response of a system which has impulse response \( f(t) \), to an impulse is \( f(t) \)"

\[ \delta(t) * f(t) = f(t) \]

"The response of a system which has impulse response \( g(t) = \delta(t) \), to an input \( f(t) \) is \( f(t) \)"
Properties of Convolution

Commutative Property:

\[ f(t) * g(t) = g(t) * f(t) \]

Proof:

\[ f * g = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \, d\tau \]

Let \( \tau_2 = t-\tau \). Then

\[ d\tau_2 = -d\tau \quad (t = \text{const. in integral}) \]

\[ \Rightarrow f * g = -\int_{-\infty}^{\infty} f(\tau_2) g(t-\tau_2) d\tau_2 \]

\[ = \int_{-\infty}^{\infty} g(t-\tau)f(\tau) \, d\tau \]

\[ = g * f \quad \text{Q.E.D.} \]

Block Diagram Interpretation:

\[ \begin{align*}
\delta(t) & \xrightarrow{G} g(t) \xrightarrow{F} f(t) * g(t) \\
\delta(t) & \xrightarrow{F} f(t) \xrightarrow{G} g(t) * f(t)
\end{align*} \]
So order of blocks is unimportant.

**Associative Property:**

\[(f * g) * h = f * (g * h)\]

**Proof** Write out two double integrals, change order of integration, show that they are the same (ugh!).

Or, use block diagram argument:

\[
\begin{align*}
\delta(t) & \quad H \quad h(t) \quad G \quad g * h \quad F \quad f * (g * h)
\end{align*}
\]

Same system:

\[
\begin{align*}
\delta(t) & \quad H \quad h(t) \quad G \quad (f * g) * h
\end{align*}
\]

\[\text{Impulse response is } f(t) * g(t)\]

Because convolution is associative, we can write \[f * g * h\] without any parentheses.