Analysis of Systems Using Laplace Transforms

Great thing about LTs is that they solve for both particular and homogeneous solutions at the same time, using only algebra!

Example Find the response of the system

\[ R = 2 \Omega \quad C = 0.1 \text{F} \]

to a step input,

\[ u(t) = 1 \text{V} \cdot u(t) \]

with initial condition

\[ y(0) = 2 \text{V} \]

[This is a mixed problem, with both an input and nonzero IC's]

To solve, use the node method to write the differential equation for the system:
\( C' e_1 + \frac{1}{R} (e_1 - u) = 0 \)

\[ \Rightarrow \quad e_1 + \frac{1}{RC} e_1 = \frac{1}{RC} u = \frac{1}{RC} \delta(t) \]

Now, simply LT both sides:

\[ sE_1(s) - e_1(0) + \frac{1}{RC} E_1(s) = \frac{1}{RC} \frac{1}{s} \]

Solve for \( E_1(s) \):

\[ \int [s + \frac{1}{RC}] E_1(s) = e_1(0) + \frac{1}{RCs} \]

\[ E_1(s) = \frac{e_1(0)}{s + \frac{1}{RC}} + \frac{1}{(s + \frac{1}{RC})(RCs)} \]

Plug in values:

\[ E_1(s) = \frac{2}{s + 5} + \frac{1}{(s + 5)(0.2s)} \]

Do a "partial fraction expansion" (more next lecture):

\[ E_1(s) = \frac{1}{s + 5} + \frac{1}{s} \quad \text{(check this!)} \]
So,

\[ L[y(t)] = L[e_1(t)] \]

\[ = \frac{1}{s + 5} + \frac{1}{s} \]

What is region of convergence?

Why? Region of convergence for unilateral transform must be of form

\[ \text{Re}[s] > \sigma_0 \]

That is, if \( \int_0^\infty e^{-st} g(t) \, dt \) converges for some \( s_1 \), it must converge for \( s_2 \), if \( \text{Re}[s_2] > \text{Re}[s_1] \), since

\[ |e^{-s_2 t}| = e^{-\text{Re}[s_2] t} \leq e^{-\text{Re}[s_1] t} = |e^{-s_1 t}| \]

\( (t > 0) \)
So, \( y(t) = L^{-1}\left[\frac{1}{s+5} + \frac{1}{s}\right] \)

\[ = \left[ e^{-5t} + 1 \right] \sigma(t) \]

(We can recognize these transforms by inspection)

So,

\[ y(t) = 1 + e^{-5t}, \quad t \geq 0 \]

\[ \text{homogeneous solution} \]

\[ \text{particular solution.} \]

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**Example** Find the impulse response of

\[ R_1 = R_3 = 252 \]

\[ C_2 = 0.4 \text{ F} \]

\[ C_4 = 0.1 \text{ F} \]
Use impedance methods to find $G(s)$; 

$$g(t) = L^{-1} \left[ G(s) \right] = \text{impulse response}$$

Circuit is a voltage divider:

$$G(s) = \frac{Z_1 || Z_2}{Z_1 || Z_2 + Z_3 || Z_4}$$

$$Z_1 || Z_2 = R_1 || \frac{1}{C_2 s}$$

$$= \frac{R_1}{C_2 s}$$

$$= \frac{R_1}{R_1 + \frac{1}{C_2 s}}$$

$$= \frac{R_1}{R_1 C_2 s + 1} = \frac{Z}{0.8 s + 1}$$

$$Z_3 || Z_4 = \frac{2}{0.2 s + 1}$$

$$\Rightarrow G(s) = \frac{2}{0.8 s + 1} \quad \frac{0.2 s + 1}{s + 2} \quad \text{(check this!)}$$
Do partial fraction expansion:

\[ G(s) = 0.2 + \frac{0.6}{s+2}, \quad \text{(check this!)} \]

\[ \Re[s] > -2 \quad \text{(why?)} \]

So

\[ g(t) = \mathcal{L}^{-1}[G(s)] = 0.2\delta(t) + 0.6e^{-2t} \]

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**Diagram:**
- A step function \( 0.2\delta(t) \) at \( t = 0 \).
- An exponential function \( 0.6e^{-2t} \) with a time constant of 2.

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