Partial Fraction Expansions

In Lecture 510, we found the inverse Laplace transform "by inspection" — we recognized the ILT of

\[ G(s) = \frac{b}{s-a}, \quad \text{Re}[s] > a \]

as \( g(t) = b e^{-at} \delta(t) \). But, we started off with LTs not in this form, e.g.,

\[ G(s) = \frac{5s+12}{s^2 + 5s + 6} \]

If we can express this \( G(s) \) as a sum of first-order poles, we can easily find the inverse Laplace transform.

How can we do this? Partial Fraction Expansion

\[ G(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0} \]

\[ = N(s)/D(s) \]
Easiest case:
- \( n > m \) (\( G(s) \) is "strictly proper")
- \( D(s) = (s-p_1)(s-p_2) \cdots (s-p_n) \)
  with distinct \( p_i \) \( [p_i \neq p_j] \)

Then we can always write

\[
G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \ldots + \frac{c_n}{s-p_n}
\]

Trick is to find \( c_i \).

**Example** \( G(s) = \frac{5s + 12}{s^2 + 5s + 6} \)

Find \( p_i \) by solving \( D(s) = 0 \):

\[
D(s) = s^2 + 5s + 6 = 0
\]

\[\Rightarrow s = -2, -3 \quad (p_1, p_2)\]

\[\Rightarrow D(s) = (s+2)(s+3)\]

So,

\[
G(s) = \frac{c_1}{s+2} + \frac{c_2}{s+3}
\]

How do we solve for \( c_1, c_2 \)?
Bad way:

\[ G(s) = \frac{C_1(s+3) + C_2(s+2)}{(s+3)(s+2)} \]

\[ = \frac{(C_1 + C_2)s + (3C_1 + 2C_2)}{s^2 + 5s + 6} \]

\[ = \frac{5s + 12}{s^2 + 5s + 6} \]

\( \Rightarrow \quad C_1 + C_2 = 5 \quad \left\{ \begin{array}{l} \text{solve for } C_1, C_2 \\ 3C_1 + 2C_2 = 12 \end{array} \right. \)

\[ C_1 = \begin{vmatrix} 5 & 1 \\ 12 & 2 \end{vmatrix} = \frac{13 - 12}{2 - 3} = 2 \]

\[ C_2 = \begin{vmatrix} 3 & 12 \\ 1 & 1 \end{vmatrix} = \frac{12 - 15}{2 - 3} = 3 \]

So \[ G(s) = \frac{2}{s+2} + \frac{3}{s+3} \]
The Cover Up Method (the right way)

Note that

\[(s - p_1) G(s) = (s - p_1) \frac{c_1}{(s - p_1)} + (s - p_1) \frac{c_2}{(s - p_2)} + \ldots\]

At \( s = p_1 \),

\[\lim_{s \to p_1} (s - p_1) G(s) = (s - p_1) \frac{c_1}{(s - p_1)} + (s - p_1) \frac{c_2}{(s - p_2)} + \ldots \]

So,

\[c_1 = \lim_{s \to p_1} (s - p_1) G(s)\]

and similarly for \( c_2, c_3, \ldots \)

Example

\[G(s) = \frac{5s + 12}{s^2 + 5s + 6} = \frac{5s + 12}{(s + 2)(s + 3)}\]

\[c_1 = (s + 2) G(s) \bigg|_{s = -2} = \frac{5s + 12}{s + 3} \bigg|_{s = -2} = \frac{2}{1} = 2\]

\[c_1 = 2\]
\[ C_2 = \frac{5s + 12}{(s + 2)(s + 3)} \bigg| _{s = -3} = \frac{-3}{-1} = 3 \]

\[ C_2 = 3 \]

just "cover up" this term

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**Example** Find the inverse LT of the unilateral LT

\[ G(s) = \frac{1}{s^2 + 2s + 5} \]

\[ s^2 + 2s + 5 = 0 \quad \Rightarrow \quad s = -1 \pm \frac{\sqrt{4 - 20}}{2} \]

\[ = -1 \pm j \cdot 2 \]

\[ \Rightarrow G(s) = \frac{1}{[s - (-1 + 2j)][s - (-1 - 2j)]} \]

\[ = \frac{C_1}{s - (-1 + 2j)} + \frac{C_2}{s - (-1 - 2j)} \]

Use cover up method to find \( C_1, C_2 \)

\[ C_1 = \frac{1}{-1 + 2j - (-1 - 2j)} = \frac{1}{4j} \]
\[ C_2 = \frac{1}{-1-2j - (-1+2j)} = -\frac{1}{4j} \]

\[ \Rightarrow G(s) = \frac{\frac{1}{4j}}{s - (-1+2j)} - \frac{\frac{1}{4j}}{s - (-1-2j)} \]

Then the inverse LT is

\[ g(t) = \frac{1}{4j} e^{(-1+2j)t} - \frac{1}{4j} e^{(-1-2j)t} \]

\[ = \frac{1}{4j} e^{-t} (e^{2jt} - e^{-2jt}) \]

\[ = \frac{1}{4j} e^{-t} \left( [\cos 2t + j \sin 2t] - [\cos 2t - j \sin 2t] \right) \]

\[ = \frac{1}{2} \sin 2t e^{-t}, \quad t \geq 0 \]

\[ g(t) = \begin{cases} \frac{1}{2} \sin 2t e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \]

This is easily verified, using LT tables.
Note: Method discussed today is general, except for cases with:

- Repeated poles, or
- \( m \geq n \) (as many "zeros" as "poles")

Will handle these cases next time.
Partial Fraction Expansion (continued)

Numerator Order $\geq$ Denominator Order

What happens when $m > n$?

$m = n$:

Example \[ G(s) = \frac{2s^2 + 6s + 5}{s^2 + 3s + 2} \]

\[ = \frac{2s^2 + 6s + 5}{(s+1)(s+2)} \]

$G(s)$ can be expanded as

\[ G(s) = C_0 + \frac{C_1}{s+1} + \frac{C_2}{s+2} \]

What is $C_0$? Equations above must be true as $s \to \infty$.

\[ \lim_{s \to \infty} G(s) = \frac{2}{1} = 2 \quad (s^2 \gg s, s^2 \gg 1) \]

Or, use L'Hôpital's rule. Also,

\[ \lim_{s \to \infty} G(s) = C_0 + \frac{C_1}{\infty} + \frac{C_2}{\infty} = C_0 \]
That is, the constant term is the ratio of the coefficient of the leading term in the numerator to the coefficient of the leading term in the denominator.

Remaining terms can be found by cover-up method:

\[ C_1 = \frac{2s^2 + 6s + 5}{s + 2} \bigg|_{s = -1} = \frac{1}{1} = 1 \]

\[ C_2 = \frac{2s^2 + 6s + 5}{s + 1} \bigg|_{s = -2} = \frac{1}{-1} = -1 \]

\[ \Rightarrow G(s) = 2 + \frac{1}{s + 1} - \frac{1}{s + 2} \]

\[ \Rightarrow g(t) = 2 \delta(t) + e^{-t} \sigma(t) - e^{-2t} \tau(t) \]

\( m > n; \)

Example

\[ G(s) = \frac{2s^2 + 5s + 5}{s + 1} \]

\[ = c_0 s + c_1 + \frac{C_2}{s + 1} \]
Find constants by long division:

\[
\begin{array}{c}
\phantom{2s+3} \\
2s+3 \\
\hline
s+1 | 2s^2 + 5s + 5 \\
\phantom{2s^2 + 2s} \\
2s^2 + 2s \\
\hline
3s + 5 \\
3s + 3 \\
\hline
2
\end{array}
\]

remainder.

\[ G(s) = 2s + 3 + \frac{2}{s+1} \]

[For higher order problem, would need to do coverup method on either \( G(s) \) or remainder term]

\[ g(t) = 2 \delta(t) + 3 \delta(t) + 2e^{-t} \delta(t) \]

"doublet," which is really a differentiator.
The Laplace Transform of a Convolution

Basic result:

\[ L \left[ g(t) \ast u(t) \right] = G(s) \cdot U(s) = Y(s) \]

Two ways to show this:

1. Direct Integration

\[ Y(s) = L \left[ g(t) \ast u(t) \right] = \int_0^\infty \left[ \int_0^t g(t-\tau) u(\tau) \, d\tau \right] e^{-st} \, dt \]

\[ = \int_0^\infty \int_0^t g(t-\tau) u(\tau) e^{-st} \, d\tau \, dt \]

The region of integration is triangular:
Change variables to make integration area "square"

\[
\begin{align*}
  t' &= t - \tau' \\
  \tau &= \tau' \\
  \tau' &= \tau - \tau \\
  t &= t' + \tau'
\end{align*}
\]

Change of variables in two dimensions:

\[
\begin{pmatrix} t' \\ \tau' \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} t \\ \tau \end{pmatrix}
\]

\[
dt' \, d\tau' = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \, dt \, d\tau = dt \, d\tau
\]

(More generally, need the Jacobian of the transformation)

Therefore,

\[
Y(s) = \int_0^\infty \int_0^\infty g(t') u(\tau') e^{-s(t'+\tau')} \, dt' \, d\tau'
\]

\[
= \int_0^\infty \int_0^\infty g(t) u(\tau) e^{-st} e^{-s\tau} \, dt' \, d\tau'
\]
\[
= \int_0^\infty g(t) e^{-st} \, dt \int_0^\infty u(\tau) e^{-s\tau} \, d\tau
\]

\[
= G(s) U(s)
\]

Note: the above derivation is valid so long as each integral is absolutely convergent.

2. Use properties of linear systems:

\[
\delta(t) \quad \rightarrow \quad H \quad \rightarrow \quad h(t) \quad \rightarrow \quad G \quad \rightarrow \quad g(t) \ast h(t)
\]

So impulse response is \( g(t) \ast h(t) \).

What is transfer function?

\[
e^{st} \quad \rightarrow \quad H(s) e^{st} \quad \rightarrow \quad G(s) H(s) e^{st}
\]

So transfer function is \( G(s) H(s) \)

\[
\Rightarrow \mathcal{L}[g(t) \ast h(t)] = G(s) H(s).
\]

Easy!
Example Find the response of the circuit
to the input $u(t) = \begin{cases} e^{-t}, & t > 0 \\ 0, & t < 0 \end{cases}$

$G(s) =$ transfer function

$= \frac{2!11 \frac{1}{s}}{2!11 \frac{1}{s} + 1}$ (voltage divider)

$= \frac{2}{s} = \frac{2/s}{s + 1/s} = \frac{2}{s + 1}$

$G(s) = \frac{2}{2s + 1} = \frac{2}{2s + 1}$

$= \frac{2}{s + 3} = \frac{1}{s + 1.5}$

$\mathcal{L}[u(t)] = \frac{1}{s + 1}$
\[ y(s) = G(s)U(s) \]

\[ = \frac{1}{s+1.5} \cdot \frac{1}{s+1} \]

\[ = \frac{2}{s+1} - \frac{2}{s+1.5} \]

\[ \Rightarrow y(t) = \left[ \frac{2}{1}e^{-t} - \frac{2}{1.5}e^{-1.5t} \right] \sigma(t) \]