Unity Feedback Control With Noise

Consider a typical unity feedback control system

\[ e' \] is the error perceived by the control system; \( e \) is the actual error. The important transfer functions are

\[
\begin{align*}
\frac{Y}{R}(s) &= \frac{1}{1 + K(s)G(s)} \\
&\equiv S(s) \\
\frac{E}{D}(s) &= \frac{-1}{1 + K(s)G(s)} \\
&\equiv -S(s) \\
\frac{E}{V}(s) &= \frac{-K(s)G(s)}{1 + K(s)G(s)} \\
&= -T(s)
\end{align*}
\]

\( S(s) \) = Sensitivity transfer function

\( T(s) \) = Complementary Sensitivity transfer function

For low sensitivity to disturbances, want:

\[ |S(s)| \ll 1 \]
For good tracking of the reference input, want:

$$|S(s)| \ll 1$$

For low sensitivity to sensor noise or errors, want:

$$|T(s)| \ll 1$$

But these goals are mutually exclusive, since

$$S(s) + T(s) = 1$$

So there is a fundamental trade-off between good tracking performance and low sensitivity to sensor noise.

How is this trade-off addressed?  
In most (not all) systems, want good tracking performance at low frequency, low sensitivity to sensor noise at high frequency:

- Reference inputs are low frequency
- Sensor noise is usually high frequency

So let’s look at the lowest frequency, $\omega = 0$ ($s = 0$)...

**Steady-State Errors**

Consider a unity feedback system without sensor noise or disturbance:

![Block Diagram](image)

For stability, define

$$L(s) = K(s)G(s) = "Loop \text{ Gain}"

What is the steady-state error to a unit step input?
Use LTs:

\[ E(s) = S(s)R(s) = \frac{1}{1 + L(s)} R(s) = \frac{1}{1 + L(s)} \frac{1}{s} \]

To find the steady error, use final value theorem:

\[ \lim_{s \to 0} e(t) = \lim_{s \to 0} sE(s) = \frac{1}{1 + L(0)} \]

If L(0) is finite, we define

\[ K_p = L(0) = "positive error constant" \]

Furthermore, if L(0) is finite, we say that a system is a “type 0 system”.
So a type 0 system will always have a finite error in response to a steady input r, but the error can be made small by making the position error constant large.
To make the steady error zero, we must have that L(0) is infinite. Suppose we can express L(s) as

\[ L(s) = \frac{L_0(s)}{s} \]

where \( L_0(0) \neq 0 \), \( L_0(0) \) is finite. Then L is a “type 1 system” (one pole at \( s = 0 \)). We have that
\[ \lim_{s \to 0} e(t) = \lim_{s \to 0} s \frac{1}{1 + \frac{L_0(s)}{s}} \frac{1}{s} = \lim_{s \to 0} \frac{s}{s + L_0(s)} = 0 \]

since \( L_0(0) \neq 0 \).

What if we want to track a unit ramp instead?

\[ r(t) = tr(t) \]
\[ \Rightarrow R(s) = \frac{1}{s^2} \]

The steady-state error for a type 0 system will be

\[ e_{ss} = \lim_{s \to 0} sS(s)R(s) \]
\[ = \lim_{s \to 0} s \frac{1}{1 + L(s)} \frac{1}{s^2} \]
\[ = \lim_{s \to 0} \frac{1}{1 + L(0)} = \infty \]

The steady-state error for a type 1 system will be

\[ e_{ss} = \lim_{s \to 0} sS(s)R(s) \]
\[ = \lim_{s \to 0} s \frac{1}{1 + \frac{L_0(s)}{s}} \frac{1}{s^2} \]
\[ = \lim_{s \to 0} \frac{1}{s + L_0(s)} \]
\[ = \frac{1}{L_0(s)} \]

which is finite. We define

\[ K_v = L_0(s) = "velocity error constant" \]

More generally, suppose that \( L(s) \) has the form

\[ L(s) = \frac{L_0(s)}{s^n} \]

\( L \) is said to be a type \( n \) system, and the error constant is \( K_p, or K_v or K_a... = L_0(0). \)
\[K_p = K_0 = \lim_{s \to 0} L(s), \ n = 0\]
\[K_v = K_1 = \lim_{s \to 0} sL(s), \ n = 1\]
\[K_a = K_2 = \lim_{s \to 0} s^2L(s), \ n = 2\]
\[\vdots\]

<table>
<thead>
<tr>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
</tr>
<tr>
<td>Type 0</td>
</tr>
<tr>
<td>Type 1</td>
</tr>
<tr>
<td>Type 2</td>
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</tbody>
</table>

Obviously, this generalizes, but we usually care most about \(K_p\) and \(K_v\) - higher order inputs are rare.
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