Problem #1

Assume:
• Incompressible
• 2-D flow \( \Rightarrow V_z = 0, \frac{\partial}{\partial z} = 0 \)
• Steady \( \Rightarrow \frac{\partial}{\partial t} = 0 \)
• Parallel \( \Rightarrow V_r = 0 \)

a) Conservation of mass for a 2-D flow is:
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r V_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} (V_\theta) = 0
\]
\( \Rightarrow \frac{\partial}{\partial \theta} (V_\theta) = 0 \Rightarrow V_\theta \) does not depend on \( \theta \)
\( \Rightarrow V_\theta = V_\theta(r) \)

b) \( \theta \)-momentum equation is:
\[
\frac{\partial}{\partial \theta} (r V_r) + (\bar{V} \cdot \nabla) V_\theta + \frac{\partial}{\partial \theta} \left( \frac{1}{2} r^2 \frac{\partial V_\theta}{\partial \theta} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial \theta} + \nu (\nabla^2 V_\theta + 2 \frac{\partial^2 V_\theta}{r^2} - \frac{V_\theta}{r^2})
\]

In cylindrical coordinates:
\[
(\bar{V} \cdot \nabla) = \frac{1}{2} \frac{\partial}{\partial \theta} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta}
\]

Thus,
\[
(\bar{V} \cdot \nabla) V_\theta + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} \frac{\partial V_\theta}{\partial \theta} = 0
\]

Also,
\[
\nabla^2 V_\theta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2}
\]
Combining all of these results gives:

\[
\frac{1}{p r} \frac{\partial p}{\partial \theta} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_\theta}{\partial r} \right) - \frac{V_\theta}{r^2} \right]
\]

this side is independent of \( \theta \)

Since the right-hand-side (RHS) is independent of \( \theta \), this requires that \( \frac{\partial p}{\partial \theta} = \text{constant for fixed } r \). But as \( \theta \) varies from \( 0 \rightarrow 2\pi \), it must be equal at \( 0 \& 2\pi \), that is, \( p(\theta = 0) = p(\theta = 2\pi) \). If not, the solution would be discontinuous.

Thus, \( \frac{\partial p}{\partial \theta} = 0 \Leftrightarrow \text{constant must be zero!} \)

The differential equation for \( V_\theta \) is:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_\theta}{\partial r} \right) - \frac{V_\theta}{r^2} = 0
\]

A little rearranging gives:

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r V_\theta) \right] = 0
\]

Integrating once gives:

\[
\frac{1}{r} \frac{d}{dr} (r V_\theta) = C_1
\]

Integrating again gives:

\[
r V_\theta = \frac{1}{2} C_1 r^2 + C_2
\]

\[\Rightarrow V_\theta = \frac{1}{2} C_1 r + \frac{C_2}{r}\]
Next, we must apply the no-slip boundary conditions to find \( V_\theta \). Specifically,

at \( r = r_o \), \( V_\theta = \omega_o r_o \)

at \( r = r_i \), \( V_\theta = \omega_i r_i \)

because flow velocity equals wall velocity in a viscous flow.

So, apply \( r = r_o \) & \( r = r_i \) bc's:

\[
\begin{align*}
\omega_o r_o & = \frac{1}{2} C_1 r_o + \frac{C_2}{r_o} \\
\omega_i r_i & = \frac{1}{2} C_1 r_i + \frac{C_2}{r_i}
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\frac{1}{2} C_1 = \frac{\omega_i r_i - \omega_o r_o}{r_i - r_o} \\
C_2 = \frac{r_o r_i (\omega_o - \omega_i)}{r_i - r_o}
\end{cases}
\]

Or, rearranged a little gives:

\[
V_\theta = r_o \omega_o \frac{r_i/r - r/r_i + r_i \omega_i/r - r_o/r_o}{r_i - r_o/r_i - r_o/r_i}
\]

c) The radial momentum equation is:

\[
\frac{\partial V_\theta}{\partial t} + (\bar{V} \cdot \nabla) V_\theta - \frac{1}{r} V_\theta^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \nabla^2 V_\theta - \frac{V_r}{r^2} - \frac{2}{r^3} \frac{\partial V_\theta}{\partial \theta} \right)
\]

But \( V_r = 0 \) & \( \frac{\partial V_\theta}{\partial \theta} = 0 \) so this reduces to:

\[
\frac{\partial p}{\partial r} = \frac{\rho V_\theta^2}{r}
\]

Since \( \frac{\rho V_\theta^2}{r} \geq 0 \) always, then clearly \( \frac{\partial p}{\partial r} \geq 0 \).

Thus, pressure increases with \( r \).
d) On the inner cylinder, the moment is a result of the skin friction due to the fluid shear stress. For this flow in which only $V_\theta \neq 0$ and is only a function of $r$, the only non-zero shear stress is $\tau_{r\theta}$ and has the following form:

$$\tau_{r\theta} = \mu \left( \frac{\partial V_\theta}{\partial r} \frac{V_\theta}{r} \right) = \mu \left[ r \frac{\partial}{\partial r} \left( \frac{V_\theta}{r} \right) \right]$$

$\tau_{r\theta}$, the only non-zero strain.
Rotating Cylinders

For the problem you studied in the homework:

1. What direction is the fluid element acceleration?

2. What direction are the net pressure forces on a fluid element?

3. What direction are the net viscous forces on a fluid element?