Poiseuille Flow Through a Duct in 2-D

Assumptions:
- Velocity is independent of $x$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$
- Incompressible flow
- Constant viscosity, $\mu$
- Steady
- Pressure gradient along length of pipe is non-zero, i.e. $\frac{\partial p}{\partial x} \neq 0$

Boundary conditions:
- No slip: $u(y = \pm h) = 0$, $v(y = \pm h) = 0 \iff$ walls are not moving

To be clear, we now will take the compressible, unsteady form of the N-S equations and carefully derive the solution:

Conservation of mass:

$$ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 $$

But $\frac{\partial \rho}{\partial t} = 0$ because flow is steady and incompressible. Also, since $\rho = \text{constant}$, then $\nabla \cdot (\rho \vec{V}) = \rho \nabla \cdot \vec{V}$

$$ \Rightarrow \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 $$

Finally, $\frac{\partial u}{\partial x} = 0$ because of assumption #1 $\rightarrow$ long pipe.

$$ \Rightarrow \frac{\partial v}{\partial y} = 0 $$
Now, integrate this:

\[ \nu = \text{constant} = C \]

Apply boundary conditions: \( \nu(\pm h) = 0 \Rightarrow \nu(y) = 0 \)

We expect this but it is good to see the math confirm it.

Now, let's look at \( y \)-momentum.

Conservation of \( y \)-momentum:

\[
\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}
\]

\[
\rho \frac{\partial v}{\partial t} + \vec{p} \cdot \nabla v = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}
\]

\[
\rho \left. \frac{\partial v}{\partial x} \right|_{v=0} + \left. \frac{\partial v}{\partial x} \right|_{\nu=0} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}
\]

Now, what about \( \tau_{xy} \) & \( \tau_{yy} \)

\[
\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \rightarrow \tau_{xy} = \mu \frac{\partial u}{\partial y}
\]

\[
\tau_{yy} = 2\mu \left( \frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x} \right) \rightarrow \tau_{yy} = 0
\]

So \( y \)-momentum becomes:

\[
0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial y} \right)
\]

But \( \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial y} \right) = 0 \) because \( \frac{\partial u}{\partial x} = 0 \) & \( \mu = \text{constant} \)
\[
\frac{\partial p}{\partial y} \Rightarrow p(x, y, t) = p(x)
\]

Conservation of \(x\)-momentum:

\[
\rho \frac{Du}{Dt} = -\frac{dp}{dx} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}
\]

\[
\rho \left( \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \frac{\partial}{\partial x} \left( 2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right)
\]

Now, we just need to solve this...

\[
\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = \frac{dp}{dx}
\]

\(\mu = \text{const} \& u = u(y)\) so,

\[
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx} \Rightarrow \text{must be constant}
\]

\[
\Rightarrow \frac{dp}{dx} = \text{const.} \Rightarrow \text{pressure can only be a linear function of } x!
\]

Now, integrating twice in \(y\) gives:

\[
u(y) = \frac{1}{2\mu} \frac{dp}{dx} y^2 + C_1 y + C_0
\]

Finally, apply BC's:
\[ u(\pm h) = 0 \]
\[ u(+h) = \frac{1}{2\mu} \frac{dp}{dx} h^2 + C_1 h + C_0 = 0 \]
\[ u(-h) = \frac{1}{2\mu} \frac{dp}{dx} h^2 - C_1 h + C_0 = 0 \]

Solve for \( C_0 \) & \( C_1 \) gives:

\[ C_0 = -\frac{1}{2\mu} \frac{dp}{dx} h^2 \]
\[ C_1 = 0 \]

\[ \Rightarrow \quad u(y) = \frac{-1}{2\mu h^2} \frac{dp}{dx} \left[ 1 - \left( \frac{y}{h} \right)^2 \right] \]