Solutions of the Laminar Boundary Layer Equations

The boundary layer equations for incompressible steady flow, i.e.,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

Note: since \( \frac{\partial p}{\partial y} = 0 \), we set \( p = p_e(x) \), i.e. the boundary layer edge pressure.

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \mu \frac{\partial^2 u}{\partial y^2}
\]

have been solved for a handful of important cases. We will look at the results for a flat plate and a family of solutions called Falkner-Skan Solutions.

Flat Plate (Laminar): Blasius Solution

For a flat plate, \( p_e = p_\infty \leftarrow \text{constant} \)

\[
\Rightarrow \frac{dp_e}{dx} = 0
\]

Blasius was able to show that the boundary later equations could be rewritten to only depend on a parameter,

\[
\eta \equiv y \sqrt{\frac{V_\infty}{2v\nu}}
\]

and its derivatives

The resulting solution has been tabulated and compared to experiments on the following page. Note:

\[
u(x, y) = V_\infty f'(\eta) \quad \text{where} \quad f' = \frac{df}{d\eta}
\]

\[
\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{\nu V_\infty f'(0)}{\sqrt{2v\nu / V_\infty}}
\]

These values from the solution of \( f(\eta) \) can be used to find:

\[
\delta_{99\%} \equiv y - \text{location at which} \ u(x, y) = 0.99 V_\infty
\]
From the table, \( f'(\eta) = 0.99 \) at \( \eta \approx 3.5 \):

\[
\eta = v \sqrt{\frac{V_x}{2yx}} \\
3.5 = \delta_{99\%} \sqrt{\frac{V_x}{2yx}} \\
\Rightarrow \delta_{99\%} = 3.5 \sqrt{\frac{2yx}{V_x}} \quad \text{← boundary layer grows as } \sqrt{x}
\]

Typically, this result is written “non dimensionally” as:

\[
\frac{\delta_{99\%}}{x} = \frac{5.0}{\sqrt{\text{Re}_x}} \quad \text{where } \text{Re}_x = \frac{V_x x}{v}
\]

Reynold’s number based on \( x \)

We can also find:

\[
\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\text{Re}_x}} \\
\frac{\theta}{x} = \frac{0.664}{\sqrt{\text{Re}_x}} \\
C_f = \frac{\tau_w}{\frac{1}{2} \rho_\infty V_x^2} = \frac{0.664}{\sqrt{\text{Re}_x}}
\]

Comment:
* At leading edge of a flat plate \( x \to 0 \) and this gives \( C_f \to \infty \)!
* In reality, the leading edge of an infinitely thin plate would have very large, but not infinite \( C_f \).
* The problem is that near the leading edge of a thin plate, the boundary layer equations are not correct and the Navier-Stokes equations are needed. **Question:** Why did the boundary layer approximation fail at \( x \to 0 \)?