Derivation of Sound Wave Properties

Assume:
* Sound wave creates small disturbances in an isentropic manner.

**Mass**

\[
\frac{d}{dt} \left[ \int_0^x \rho(x) dx + \rho u \right] - \rho u_0 = 0
\]

\[
\frac{d}{dt} \left[ \int_0^x (\rho + d\rho) dx + \int_0^x \rho dx \right] + \rho u_0 - (\rho + d\rho) du = 0
\]

\[
\frac{d}{dt} \left[ \int_0^x d\rho dx + \int_0^x \rho dx \right] - (\rho + d\rho) du = 0
\]

\[
\frac{d}{dt} [d\rho x_s] - (\rho + d\rho) du = 0
\]

\[
d\rho \frac{dx_s}{dt} - (\rho + d\rho) du = 0
\]

\[
d\rho \ a - \rho du - d\rho du = 0
\]

\[
\Rightarrow \ d\rho = \rho du
\]
Isentropic disturbances

Constant entropy disturbances for perfect, ideal gases satisfy:

\[ \frac{p}{\rho^\gamma} = \text{const.} \]

\[ \Rightarrow \frac{p + dp}{(\rho + d\rho)^\gamma} = \frac{p}{\rho^\gamma} \]

\[ \Rightarrow \frac{p + dp}{p} = \left( \frac{\rho + d\rho}{\rho} \right)^\gamma \]

\[ 1 + \frac{dp}{p} = \left( 1 + \frac{d\rho}{\rho} \right)^\gamma \]

\[ 1 + \frac{dp}{p} = 1 + \gamma \frac{d\rho}{\rho} \]

\[ dp = \frac{\gamma p}{\rho} d\rho \]

Conservation of Momentum

\[ \frac{d}{dt} \left( \rho u dx + (\rho u^2 + p) \right)_i - (\rho u^2 + p)_0 = 0 \]

Higher order terms eliminated

\[ \rho du a + p - (p + dp) = 0 \]

\[ \Rightarrow \rho du a = dp \]

Summarizing:
Mass: \[ a d\rho = \rho du \] (1)
Isentropic: \[ dp = \frac{\gamma p}{\rho} d\rho \] (2)
Momentum: \[ \rho du a = dp \] (3)

Combining \( a \) * (1) − (3) gives:

\[ a^2 d\rho = dp \]

Then, using (2) gives:

\[ a^2 d\rho = \frac{\gamma p}{\rho} d\rho \]

\[ \Rightarrow \left( a^2 - \frac{\gamma p}{\rho} \right) d\rho = 0 \]

Since \( d\rho \neq 0 \), then \[ a^2 = \frac{\gamma p}{\rho} \]. We’ve just derived the speed of sound for an
Ideal, perfect gas.

**Note:** without assuming ideal, perfect gas, the general result is \( a^2 = \left. \frac{\partial p}{\partial \rho} \right|_{s=const.} \).

One other thing of interest: Suppose the sound wave caused a change in pressure, \( dp \). Then, the change in velocity is:

\[
du = \frac{1}{\rho a} dp \Rightarrow du \text{ has same sign as } dp
\]