Linearized Subsonic Flow

We desire to solve

\[ (1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \]  \hfill (1.1)

\[ 0 < M_\infty < 1 \]  \hfill (1.2)

\[ (1 - M_\infty^2) \geq 0 \]  \hfill (1.3)

In two dimensions, we have

\[ (1 - M_\infty^2) \phi_{xx} + \phi_{yy} = 0 \]  \hfill (1.4)

\[ v' = U_\infty \left( \frac{\partial y}{\partial x} \right)_{\text{BODY}} \]  \hfill (1.5)

\[ u' \to 0, \quad x, y \to \infty \]  \hfill (1.6)

\[ v' \to 0, \quad x, y \to \infty \]  \hfill (1.7)

Now let

\[ \beta = \sqrt{1 - M_\infty^2} \]  \hfill (1.8)

and transform independent coordinates as follows:

\[ \xi = x \]  \hfill (1.9)

\[ \eta = \beta y \]  \hfill (1.10)

And likewise, the dependent perturbation velocity potential

\[ \tilde{\phi}(\xi, \eta) = \beta \phi(x, y) \]  \hfill (1.11)

This series of transformations lead to the following

\[ \frac{\partial \xi}{\partial x} = 1 \quad \frac{\partial \xi}{\partial y} = 0 \quad \frac{\partial \eta}{\partial x} = 0 \quad \frac{\partial \eta}{\partial y} = \beta \]  \hfill (1.12)

\[ \phi_x = \frac{\partial \phi}{\partial x} = \frac{1}{\beta} \frac{\partial \tilde{\phi}}{\partial x} + \frac{1}{\beta} \left[ \frac{\partial \tilde{\phi}}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{\phi}}{\partial \eta} \frac{\partial \eta}{\partial x} \right] = \frac{1}{\beta} \frac{\partial \tilde{\phi}}{\partial \xi} = \frac{1}{\beta} \tilde{\phi}_{\xi} \]  \hfill (1.13)

\[ \phi_{xx} = \frac{1}{\beta} \phi_{\xi \xi} \]  \hfill (1.14)
\[
\phi_y = \frac{\partial \phi}{\partial y} = \frac{1}{\beta} \frac{\partial \tilde{\phi}}{\partial y} = \frac{1}{\beta} \left[ \frac{\partial \tilde{\phi}}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{\phi}}{\partial \eta} \frac{\partial \eta}{\partial y} \right] = \frac{\partial \tilde{\phi}}{\partial \eta} = \tilde{\phi}_\eta
\]

(1.15)

\[
\phi_{yy} = \beta \tilde{\phi}_\eta
\]

(1.16)

Our transformed governing equation becomes

\[
\beta^2 \left( \frac{1}{\rho} \tilde{\phi}_{\xi \xi} \right) + \beta \tilde{\phi}_{\eta \eta} = 0
\]

(1.17)

or

\[
\tilde{\phi}_{\xi \xi} + \tilde{\phi}_{\eta \eta} = 0
\]

(1.18)

Our analysis drives us to the following question:

**How can we exploit incompressible results to account for compressibility effects?**

Compare the forms

\[
\phi_{xx} + \phi_{yy} = 0, \quad M_\infty = 0
\]

(1.19)

\[
\tilde{\phi}_{\xi \xi} + \tilde{\phi}_{\eta \eta} = 0, \quad \beta > 0
\]

(1.20)

Consider the boundary condition on the airfoil surface.

\[
y = f(x), \quad \text{airfoil shape in } x, y
\]

(1.21)

\[
\eta = q(\xi), \quad \text{airfoil shape in } \xi, \eta
\]

(1.22)

Our boundary condition may be expressed as follows:

\[
U_\infty \frac{df}{dx} = \frac{\partial \phi}{\partial y} = \frac{1}{\beta} \frac{\partial \tilde{\phi}}{\partial y} = \frac{\partial \tilde{\phi}}{\partial \eta}
\]

(1.23)

\( (x, y) \) space

Similarly in \( (\xi, \eta) \) space

\[
U_\infty \frac{dq}{d\xi} = \frac{\partial \tilde{\phi}}{\partial \eta}
\]

(1.24)

\( (\xi, \eta) \) space

Therefore,

\[
U_\infty \frac{df}{dx} = U_\infty \frac{dq}{d\xi}
\]

(1.25)

or

\[
\frac{df}{dx} = \frac{dq}{d\xi}
\]

(1.26)

Conclusions

(a) The shape of the airfoil in \( x, y \) space is the same in \( \xi, \eta \) space.

(b) \( \tilde{\phi}, \xi, \eta \) implies that the compressible flow over an airfoil in \( x, y \) space is related to an incompressible flow in \( \xi, \eta \) space over the same airfoil.

Now, let’s return to the pressure coefficient:

\[
\frac{c_p}{U_\infty} = -2 \frac{U_\infty}{U_\infty} \frac{1}{U_\infty} \frac{\partial \phi}{\partial x}
\]

\[
= - \frac{1}{U_\infty} \frac{\partial \phi}{\partial x}
\]

\[
= - \frac{1}{U_\infty} \frac{\partial \tilde{\phi}}{\partial \xi}
\]

(1.27)
Let the incompressible pressure coefficient be

\[ c_{p_0} \equiv -2 \frac{\ddot{u}}{U_\infty} = -2 \frac{1}{U_\infty} \frac{\partial \Phi}{\partial \xi} \]  

(1.28)

Therefore, substituting

\[ c_p = \frac{1}{\beta} c_{p_0} \]  

(1.29)

\[ c_p = \frac{c_{p_0}}{\sqrt{1 - M_\infty^2}} \]  

(1.30)

This is the Prandtl-Glauert rule. It is a similarity rule that relates incompressible flow over a given two-dimensional profile to subsonic compressible flow over the same profile.

From above results, it can be shown that

\[ c_L = \frac{c_{L_0}}{\sqrt{1 - M_\infty^2}} \]  

(1.31)

\[ C_M = \frac{C_{M_0}}{\sqrt{1 - M_\infty^2}} \]  

(1.32)

\[ u' = \frac{\ddot{u}}{\sqrt{1 - M_\infty^2}} \]  

(1.33)

What does it mean?