Steady laminar boundary layer flow when subject to small disturbances may become unstable (above a critical Reynolds) change flow to turbulent flow. We would like to examine stability of the flow subject to small perturbations. Will they grow? Unstable, or decay - stable.

Ref to previous stability analyses, introduce small perturbation on mean flow.

\[ \nabla \cdot \bar{U}_0 = 0 \]

\[ \frac{\partial \bar{U}_0}{\partial t} - \nabla p_0 + \frac{1}{Re} \nabla^2 \bar{U}_0 \]

Is mean flow stable.
Is small disturbance.

\[ \bar{U} = \bar{U}_0 + \hat{U} \]

\[ \hat{p} = p_0 + \hat{p} \]

where \(|\hat{U}| \ll |\bar{U}_0|\)

Substitute above and neglect higher powers of \(\hat{U}\) & \(\hat{p}\)

\[ \nabla \cdot \hat{U} = 0 \]

\[ \frac{\partial \hat{U}}{\partial t} + U_0 \frac{\partial \hat{U}}{\partial x} + \hat{U} \frac{\partial U_0}{\partial x} + V_0 \frac{\partial \hat{U}}{\partial y} + \hat{U} \frac{\partial U_0}{\partial y} + \frac{\partial \hat{p}}{\partial x} + \hat{U} \frac{\partial U_0}{\partial z} = - \frac{\partial \hat{p}}{\partial x} + \frac{1}{Re} \nabla^2 \hat{U} \]

- linear PDE
Assume the perturbed solution has the form

$$
\hat{u} = \hat{u}(y) e^{i(\alpha x + \beta z - \omega t)} \quad \hat{v} = \hat{v}(y) e^{i(\alpha x + \beta z - \omega t)}
$$

Assume for simplicity that $\hat{u}_0 = (u_0(y), 0, 0)$ – 2D parallel flow
- approximate for Falkner-Skan flow $\frac{\partial u}{\partial x} = 0$
- exact for Poiseuille flow

Linearized cont. + mom. simplifies to

$$
\frac{\partial \hat{u}}{\partial x} + 2 \frac{\partial \hat{v}}{\partial y} + \alpha \frac{\partial \hat{w}}{\partial z} = 0
$$

$$
\frac{\partial \hat{u}}{\partial t} + u_0 \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial \hat{u}}{\partial y} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re} \nabla^2 \hat{u}
$$

$$
\frac{\partial \hat{v}}{\partial t} + u_0 \frac{\partial \hat{v}}{\partial x} = -\frac{\partial \hat{p}}{\partial y} + \frac{1}{Re} \nabla^2 \hat{v}
$$

$$
\frac{\partial \hat{w}}{\partial t} + u_0 \frac{\partial \hat{w}}{\partial x} = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{Re} \nabla^2 \hat{w}
$$

Substitute $\hat{u}(y) e^{i\xi (\cdot)}$

Note: $\frac{\partial}{\partial x} = i \xi$, $\frac{\partial}{\partial z} = i \beta$, $\frac{\partial}{\partial t} = i \omega$

$$
\Rightarrow \quad i \xi \hat{u} + \frac{\partial \hat{v}}{\partial y} + i \beta \hat{w} = 0
$$

$$
-i \omega \hat{u} + i \alpha u_0 \hat{u} + \frac{\partial u_0 \hat{v}}{\partial y} = -i \xi \hat{p} + \frac{1}{Re} \left( \frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2 \right) \hat{u}
$$

$$
-i \omega \hat{v} + i \alpha u_0 \hat{v} = - \frac{\partial \hat{p}}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2 \right) \hat{v}
$$

$$
-i \omega \hat{w} + i \alpha u_0 \hat{w} = -i \beta \hat{p} + \frac{1}{Re} \left( \hat{v} \right)
$$
let $\beta > 0 \Rightarrow \omega = 0$ (discontinuity propagates in flow direction)

- Squire's theorem: worst case - lowest Re.


\[
\begin{align*}
\omega &= \frac{i \alpha}{k} \frac{\partial \psi}{\partial y} \\
\phi &= \frac{\alpha}{k} \frac{\partial \psi}{\partial y} \\
\eta &= \psi \\
\end{align*}
\]

At critical Reynolds $Re$, unstable waves correspond to $\beta = 0$.

2D disturbance has adequate to understand and model for practical engineering cases.

From continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\tilde{u} = \frac{1}{\nu} \frac{\partial \psi}{\partial y}
\]

Squire's transform:

\[
\tilde{\alpha}' = \frac{2}{\sqrt{\nu + \rho' \nu}} \quad \quad \tilde{\beta}' = \frac{\alpha \nu}{\sqrt{\nu + \rho' \nu}}
\]

Remove dependence on $\beta$.

\[\begin{align*}
&x - \text{mom} \rightarrow \frac{1}{\nu} \left[ \frac{\partial u_0}{\partial y} - \frac{i}{\nu} (\alpha u_0 - \omega) \frac{\partial \psi}{\partial y} - \frac{1}{\nu} \frac{\partial (\frac{\partial^2 u_0}{\partial y^2} - \alpha^2) \frac{\partial \psi}{\partial y}}{\partial y} \right] \\
&y - \text{mom} \rightarrow \frac{\partial (\alpha u_0 - \omega) v}{\partial y} = -\frac{\partial \rho}{\partial y} + \frac{i}{\nu} \frac{\partial^2 u_0}{\partial y^2} - \frac{1}{\nu} \frac{\partial (\frac{\partial^2 u_0}{\partial y^2} - \alpha^2) \frac{\partial \psi}{\partial y}}{\partial y}
\end{align*}\]

Simplify,

\[\begin{align*}
\Rightarrow (\alpha u_0 - \omega) \left( \frac{\partial v}{\partial y} - \alpha^2 v \right) - \nu \frac{\partial^2 u_0}{\partial y^2} v + \frac{i}{\nu} \left( \frac{\partial^2 u_0}{\partial y^2} - 2 \alpha^2 \frac{\partial^2 v}{\partial y^2} + \alpha^4 \right) v &= 0
\end{align*}\]

- Orr-Sommerfeld equation - 4th order ODE for $\psi(y)$

- $U(y)$ is input (mean flow)

- $w = \alpha c$ - wave number, $c$ wave speed

Boundary conditions
Boundary Conditions

Duct: \( y = 0 \quad \vec{V} \cdot \vec{V}' = 0 \)
\( y = 1 \quad \vec{V} = \vec{V}' = 0 \) (Poisson flow)

B-L: \( y = 0 \quad \vec{V} = \vec{V}' = 0 \)
\( y = +\infty \quad \vec{V} = \vec{V}' = 0 \)

Free shear layer: \( y = \pm \infty \quad * = 0 \).

Note: Governing ODE and boundary conditions are homogeneous

\[ \Rightarrow \text{Eigenvalue problem: non-trivial } \vec{V}(y) \text{ exist only for certain combinations of } \mu, \nu, \text{ Re} \]

for a given mean flow \( \vec{u}(y) \)

Example: Analogous to beam buckling (instability)

\[ y'' + \frac{P}{EI} y = 0 \quad y(0) = y(L) = 0 \]

Solution: \( y = A \sin(kx) + B \cos(kx) \) - eigenfunctions

where \( k^2 = \frac{P}{EI} \) \( y > 0 \) only if \( k = \pm 1, \pm 2, \ldots \)

\( \text{eigenvalues} \quad A \text{ is arbitrary.} \)

Can only predict if unstable or not

\[
\left. \begin{array}{c}
\frac{k^2 L^2}{EI} \Rightarrow \frac{n^2 \pi^2 \mu^2 E}{L^2} = \frac{P}{EI} \\
\frac{n^2 \pi^2 \nu^2 E}{L^2} = \frac{P}{EI} \\
\frac{n^2 \pi^2 \nu E}{L^2} \leq \frac{\pi^2 \nu E}{L^2} \quad \text{(Cantor (buckling)limit)}
\end{array} \right.
\]
In general, either $u$ or $w$ will be complex.
Two types of eigenvalue problems:

a) Temporal Amplification
b) Spatial Amplification

a) Temporal Amplification:

$w$ is real and specified

$w = w_r + i w_i$ will be calculated

$e^{i w t} = e^{-i w t} e^{i w t}$

Temporal growth rate

$\omega_i > 0$ - growth in time

$\omega_i < 0$ - decay

Solve $\mathbf{L} \cdot \mathbf{v} = \mathbf{0}$ for $\mathbf{v}$, $\xi_j$ = 0

Temporal analysis predicts behavior as $t \to \infty$ given initial

perturbation / disturbance of $w_r, w_i, \xi_j$ = 0.

b) Spatial Amplification:

$w$ real is specified and $\omega = \xi_r + i \xi_i$

$\xi_r$ $\xi_i$ will be calculated

$e^{i \xi x} = e^{-i \xi x} e^{i \xi_x}$

Spatial growth rate

$\xi_r < 0$ - growth downstream

$\xi_r > 0$ - decay

Better/more relevant is BL problem where
free stream turbulence provides steady
perturbation / disturbance.
\textbf{Inviscid limit}

As \( Re \to \infty \), we get Rayleigh Eqn.

\[(\alpha U_0 - \omega)(\ddot{v}'' - 2\ddot{v}') - \alpha U_0 \ddot{v} = 0 \quad \text{(2nd order ODE)}\]

\text{Boundary conditions:} 
\[y = 0 \quad \ddot{v} = 0 \quad \text{2BCs}\]
\[y = \infty \quad \ddot{v} = 0 \]

Examine when \textit{instability} can occur in inviscid limit.

Assume \textit{temporal} problem: \( \alpha = \alpha r \) given, \( \omega = \omega + i \omega_i \)

\[
\begin{bmatrix}
\ddot{v}'' &=& \alpha r \ddot{v} + \frac{\alpha r U_0'' \ddot{v}}{\alpha r U_0 - \omega} \\
\ddot{v}'' &=& \alpha r \ddot{v} + \frac{\alpha r U_0'' \ddot{v}^*}{\alpha r U_0 - \omega^*}
\end{bmatrix} \Rightarrow \ddot{v}^* \quad \text{complex conjugate}
\]

\[
\Rightarrow \ddot{v} \dddot{v}^* - \dddot{v}'' \ddot{v}^* = \alpha r U_0'' / \dddot{v}^* \frac{1}{\alpha r U_0 - \omega} - \frac{1}{\alpha r U_0 - \omega^*}
\]

\[
\int_0^\infty \frac{d}{dy} (\ddot{v} \dddot{v}^* - \dddot{v}'' \ddot{v}) dy = \int_0^\infty \frac{\alpha r U_0'' / \dddot{v}^*}{\alpha r U_0 - \omega} 2i \omega_i dy \]

\[
0 = \omega_i \int_0^\infty \frac{U_0'' (\dddot{v})}{|U_0 - \omega |^{1/2}} dy
\]

\text{Instability} \((\omega_i > 0, |\dddot{v}|^2 \neq 0)\) possible only if \( U_0'' \) changes sign.