3) Thin Shear Layer Approximation

- A) Re → ∞ behavior
  - Ordering
  - TSL approximation

Reading: White 218-219, 227-233
  Sch. (see New ed.)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad - x \text{ comp.} \]

\[ \text{Re} = \frac{U_\infty L}{\nu} \gg 1 \]

At A, 0, 3, and 3 balance

b, 0 & 2 → 0 3 → 0 (3)

↑ do likewise

* Using \( \rho, U_\infty, v, L \) as scales, the governing equations are:

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} \]

\[ \text{Re} = \frac{U_\infty L}{\nu} \]

\[ \mathbf{u} = 0 \text{ at wall (no slip)} \]

Typical Re values are large

Example

- Pigeon = 50k
- Ant, Lenna = 5 mm
- 747 = 100 mm
- Super tanker = 5 bill
This suggests that $Re$ is a small parameter, and seek solutions as an asymptotic expansion in $\varepsilon$ ($(\varepsilon Re)^{\nu}$).

$$
\hat{u} = \hat{u}_0 + \varepsilon \hat{u}_1 + \varepsilon^2 \hat{u}_2 + \cdots
$$

$$
p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots
$$

Look first at $\hat{u}_0$, $p_0$: put (*) in NS eqns. and B.Cs.

$$
\nabla \cdot \hat{u}_0 = 0
$$

$$
d\hat{u}_0 \over dt + \hat{u}_0 \cdot \nabla \hat{u}_0 = -\nabla p_0
$$

B.C: $\hat{u}_0 = 0$

Problem: Cannot satisfy both

$u_0 = 0$ & $V_0 = 0$ at wall

only $\hat{u}_0 \cdot \hat{n} = 0$

We lost highest-order term $\varepsilon^2 \nabla^2 \hat{u}$

$\rightarrow$ singular perturbation problem.

No slip B.C forces $\varepsilon^2 \nabla^2 \hat{u}$ to be finite as $\varepsilon \to 0$

The $\parallel \parallel$ is to seek scales other than $u_0$, $L$ near wall region.

Example: In Rayleigh case, we had $\delta(x) = \sqrt{yt}$; $Y = y/\delta(t)$

In B-L case, look for $\delta(x)$ for scaling in limit $y \to L$ for $y$.

In the above problem we can switch the $y$ coordinate:

$$Y = y/e$$

$s.t.$ $Y = O(1)$ as $\varepsilon \to 0$

Near the wall we can use

$$u = u_1 + \varepsilon u_2 + \cdots$$

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

$$p = p_0 + \varepsilon p_1 + \cdots$$

$$x \text{ comp. } \Rightarrow \frac{\partial u_1}{\partial x} + \varepsilon \frac{\partial u_1}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\varepsilon^2 Re} \cdot \frac{\partial^2 u_1}{\partial y^2}

\rightarrow \varepsilon = \frac{1}{\sqrt{Re}}$$
Simple linear Orr-Sommerfeld equation with boundary conditions:

\[ \begin{align*}
  \frac{d^2 f}{dx^2} + \frac{df}{dx} &= a, \\
  f(0) &= 0, \\
  f(1) &= 1
\end{align*} \]

Exact solution:

\[ f(x; \epsilon) = (1-a) \left( \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \right) + ax \]

Setting \( \epsilon = 0 \) gives:

\[ \frac{df}{dx} = a \]

which can only satisfy the boundary condition unless \( a = 1 \):

\[ f(x; \epsilon) = (1-a) + ax \quad \text{(result of dropping highest derivative)} \]

As \( \epsilon \to 0 \):

Choose a different scale when \( x \) is small or close to the wall:

\[ x = \frac{x}{\epsilon} \longrightarrow F(x; \epsilon) \]

Substituting gives:

\[ \begin{align*}
  \frac{d^2 F}{dx^2} + \frac{dF}{dx} &= a\epsilon, \\
  F(0) &= 0, \\
  F(1/\epsilon) &= 1
\end{align*} \]

\[ f(x; \epsilon) = (1-a)(1 - e^{x/\epsilon}) \quad \text{as} \quad \epsilon \to 0 \]

but \( x = O(1) \)

Ref: Van Dyke, Perturbation Methods in Fluid Dynamics
Examine the order of magnitude of each term in governing eqn.

1) Continuity:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
\[ \begin{align*}
0 \delta \frac{U_0}{x} & \Rightarrow \frac{\delta}{x} = O \left( \frac{v}{U_0 x} \right) \\
0 \frac{1}{1} & \Rightarrow x = O(1)
\end{align*} \]

2) X-momentum:
\[ \begin{align*}
\frac{u \frac{\partial u}{\partial x}}{x} + v \frac{\partial u}{\partial y} & = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \\
& = 0 \left( \frac{U_0^2}{x} \right) \\
& \Rightarrow \frac{\delta}{x} = O \left( \frac{v}{U_0 x} \right)
\end{align*} \]

Also:
\[ \frac{U_0}{x} = O \left( \frac{v}{U_0 x} \right) \Rightarrow \frac{\delta}{x} = O \left( \frac{v}{U_0 x} \right) \]

3) Y momentum:
\[ \begin{align*}
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} & = -\frac{i}{\rho} \frac{\partial p}{\partial y} + \nu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \\
& = 0 \left( \frac{U_0^2}{x^2} \right) \\
& \Rightarrow \frac{\delta}{x} = O \left( \frac{v}{U_0 x} \right)
\end{align*} \]
This suggests
\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = O \left( \frac{u_0^2}{x^2} \delta \right) \text{ or } O(\delta)
\]

Curved wall
\[\Rightarrow \frac{1}{\rho} \frac{\partial p}{\partial y} = O \left( \frac{u_0^2}{R} \right)\]

Change in pressure across $\delta$:
\[
p(\delta) - p(0) = \Delta p = \frac{\partial p}{\partial y} \delta = O \left( \rho u_0^2 \left( \frac{\delta}{R} \right)^2 \right) \text{ or } O(\delta^2)
\]
\[\text{or } = O \left( \rho u_0^2 \left( \frac{\delta}{R} \right) \right) \text{ or } O(\delta/R) \]

which is bigger in most cases.

In summary,

Keeping terms of $O(1)$ in $x$-momentum, and $\frac{\partial p}{\partial y} = 0$ in $y$-momentum gives us **TSE Equations**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
\[

\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

\[
\frac{\partial p}{\partial y} = 0
\]

* Neglects streamline diffusion
  \[
  \frac{\partial^2 u}{\partial x^2} = 0
  \]

* Neglects normal momentum
  \[
  \frac{\partial p}{\partial y} = 0
  \]

* Assumption weakest at anfoil i.e. and
  shocks, for example...
Thick Shear Layer Approximation

Viscous flows contain 3 basic momentum transport mechanisms:

\[ \vec{u} \cdot \nabla \vec{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \vec{u} \]

These mechanisms become directionally biased in a thin shear layer:

Convection (unchanged)  
Pressure (transverse component suppressed)  
Diffusion (transverse component accentuated)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \]

\[ 0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} \]

1) Transverse velocity \( v \) is governed primarily by kinematic (continuity) requirements: \( \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \), not by dynamic (y-momentum) requirements. The y-momentum equation decouples and is neglected.

2) Streamwise diffusion is negligible compared to transverse diffusion.

In real situations, assumption 1) is weaker than 2).