Practice Problems

1. 

b) Response of system when subjected to rectangular pulse $F_0$ duration $T_p$.

The hint given in the problem suggests using the superposition principle to divide the pulse into a constant force $F_0$ applied at time 0, and a constant force $-F_0$ applied at time $T_p$.

The solution (or response) to each constant force applied individually to the system is as follows.

A. Constant force $F_0$ at time 0:

Governing equation:

\[ m \ddot{q} + kq = F_0 \]

\[ \Rightarrow \quad q(t) = \frac{F_0}{k} \left( 1 - \cos \omega t \right) - 0 \]

\[ \omega = \sqrt{\frac{k}{m}} \]

* unit #20, p.7*
B. Constant force $-F_0$ at time $T_p$:

The solution to the governing equation can be found by shifting the time frame in the solution for case A. Note that for $t < T_p$, the system is not excited, and therefore, the response is zero.

$$q = \begin{cases} 
0 & t < T_p \\
-\frac{F_0}{k} \left( 1 - \cos \omega (t - T_p) \right) & t \geq T_p
\end{cases} \quad (2)$$

Superposing equations (1) and (2), we get

$$q(t) = \begin{cases} 
\frac{F_0}{k} \left( 1 - \cos \omega t \right) & t < T_p \\
\frac{F_0}{k} \left[ -\cos \omega t + \cos \omega (t - T_p) \right] & t \geq T_p
\end{cases} \quad (3)$$

*Note: $\cos \omega t - \cos \omega \tau = 2 \sin \frac{\omega t + \omega \tau}{2} \sin \frac{\omega t - \omega \tau}{2}$

b) From equation (3) we can see that the response for $t < T_p$ is simply the response for a constant force $F_0$ applied at time $0$. 
The response for \( t \geq T_p \) is more interesting. Rewriting the response for \( t \geq T_p \),

\[
q(t) = \frac{2F_0}{k} \sin \frac{1}{2} \omega T_p \sin \frac{\omega}{2} (2t - T_p) \tag{8}
\]

\( (t \geq T_p) \)

Coefficient independent
of time

time dependent

From equation (8), we can immediately see some interesting responses for particular \( T_p \)'s due to the coefficient independent of time.

A. \( T_p = 0 \) : \[ \frac{2F_0}{k} \sin \frac{1}{2} \omega T_p = 0 \]

\( \Rightarrow \) \( q(t) = 0 \) \( \text{This is expected because if } T_p = 0, \text{ no force is being applied.} \)

B. \( T_p = \frac{\pi}{\omega} n \) : \[ \frac{2F_0}{k} \sin \frac{1}{2} \omega \left( \frac{\pi}{\omega} n \right) = \frac{2F_0}{k} \sin \frac{\pi}{2} n = 0 \]

\( \Rightarrow \) \( q(t) = 0 \) \( \text{(t } \geq T_p) \)

C. \( T_p = \frac{2\pi}{\omega} n \) : \[ \frac{2F_0}{k} \sin \frac{1}{2} \omega \left( \frac{2\pi}{\omega} n \right) = \frac{2F_0}{k} \sin \frac{\pi}{2} n \]

\( \Rightarrow \) \( q(t) = \frac{2F_0}{k} \sin \frac{\omega}{2} (2t - T_p) \sin \frac{\pi}{2} n \) \( (t \geq T_p) \)

\[
q(t) = \begin{cases} 
\frac{2F_0}{k} \sin \frac{\omega}{2} (2t - T_p) & n = 1, 5, 9, \ldots \\
- \frac{2F_0}{k} \sin \frac{\omega}{2} (2t - T_p) & n = 3, 7, 11, \ldots 
\end{cases}
\]
To plot the response, let's first normalize with respect to \( \frac{2\pi}{\omega} \), which is the period. Thus, from equation (3) \((14)\),

\[
q(t) = \begin{cases} 
\frac{F_0}{k} \left( 1 - \cos \frac{2\pi T_p}{(2\pi)^2} \left( \frac{t}{T_p} \right) \right) & \left( \frac{t}{T_p} < 1 \right) \\
\frac{2F_0}{k} \sin \pi T_p \sin \pi \left( \frac{2t}{T_p} - 1 \right) & \left( \frac{t}{T_p} \geq 1 \right)
\end{cases}
\]

If we define

\[
\frac{T_p}{(2\pi)^2} = \tau, \quad \frac{t}{T_p} = t'
\]

and normalized the response by \( F_0/k \), we get

\[
\frac{q(t)}{F_0/k} = \begin{cases} 
(1 - \cos 2\pi \tau t') & t' < 1 \\
2 \sin \pi \tau \sin \pi 2 \left( 2t' - 1 \right) & t' \geq 1
\end{cases}
\]

The plots for \( T_p/(2\pi) = 0.1, 0.2, 0.5 \) \& 1 are shown on p. 5, and for \( T_p/(2\pi) = 2 \) and 5 are shown on p. 6. For values of the ratio of pulse duration to natural frequency \( = T_p/(2\pi) \) not equal to an integer value, the mass oscillates with a frequency of \( \omega \) until \( t = T_p \), after which it oscillates with the same frequency but different amplitude ranges. The amplitude before \( t = T_p \) is between 0 and \( 2F_0/k \), while after \( t = T_p \), the amplitude is \( 2\sin \pi \tau (F_0/k) \).
For integer values of $T_p/(2\pi/5)$, the response before $t = T_p$ is the same as before (i.e., oscillates with frequency $\omega$ between 0 and $2\pi/5$), but after $t = T_p$, the response is zero. This also follows from

Case B where, for $T_p = \frac{2\pi}{5} n$ with any integer $n$, $q(t) = 0$ for $t \geq T_p$. The case for $T_p/(2\pi/5) = 10$ is not shown, but is very similar to the other integer cases.
Response for various ratios of pulse duration

- \( T_p/T = 2 \)
- \( T_p/T = 5 \)
2.

\[ E = 10.0 \text{ kips} \]
\[ \beta = 0.1 \text{ lb/ft} \]
\[ I = \frac{1}{12} \text{ in}^4 = 0.083 \text{ in}^4 \]

Represent the continuous system as a three-mass system and determine the natural frequencies and associated mode shapes. So, the first thing we have to do is to find a way to discretize the continuous beam into the three-mass system. We discussed how we could do this in unit #22 (from p. 10). The fundamental idea is to obtain the flexibility influence coefficients at each discretized point and from that, invert to obtain the stiffness influence coefficients. The flexibility influence coefficients can be obtained from

\[ C_{ij} = \frac{1}{2EI} \left( x_i^2 x_j - \frac{x_i^3}{3} \right) \]  \hspace{1cm} (unit#21, p.10)

Once we obtain the stiffness influence coefficients, we need to discretize the mass of the beam into three concentrated masses. This will allow
us to obtain a set of second-order differential equations, which are the governing equations for the three-mass system. The natural frequencies and associated mode shapes can be obtained from that equation.

Influence

a) Determine flexibility coefficients:

\[ C_{11} = \frac{1}{2EI} \left( \frac{k_1^2}{9} - \frac{1}{3} \frac{k_2^2}{27} \right) = \frac{1}{2EI} \frac{2k_1^2}{27} \]

\[ C_{21} = C_{12} = \frac{1}{2EI} \left( \frac{k_2^2}{9} \frac{2k_1}{3} \right) - \frac{1}{3} \frac{k_1^2}{27} = \frac{1}{2EI} \frac{5k_1^2}{327} = \frac{5}{162EI} l^3 \quad (3) \]

\[ C_{31} = C_{13} = \frac{1}{2EI} \left( \frac{k_3^2}{9} l - \frac{1}{3} \frac{k_1^2}{27} \right) = \frac{1}{2EI} \frac{8k_1^2}{327} = \frac{4}{327EI} l^3 \quad (4) \]

\[ C_{32} = C_{23} = \frac{1}{2EI} \left( \frac{k_2^2}{9} l - \frac{1}{3} \frac{k_1^2}{27} \right) = \frac{1}{2EI} \frac{8k_1^2}{327} = \frac{4}{327EI} l^3 \quad (5) \]

\[ C_{33} = \frac{1}{2EI} \left( l^3 - \frac{1}{3} l^3 \right) = \frac{1}{3EI} l^3 \quad (6) \]
Using symmetry, (i.e. $C_{ij} = C_{ji}$), we can write the flexibility influence coefficients in matrix form as,

$$
C = \frac{l^2}{EI} \begin{bmatrix}
\frac{1}{81} & \frac{5}{162} & \frac{4}{81} \\
\frac{5}{162} & \frac{8}{81} & \frac{14}{81} \\
\frac{4}{81} & \frac{14}{81} & \frac{1}{3}
\end{bmatrix}
$$

Plugging in the given values for $l = 60''$, $E = 10.0$ Msi, $I = 0.085in^4$,

$$
C = \begin{bmatrix}
3.20 \times 10^{-3} & 8.00 \times 10^{-3} & 1.30 \times 10^{-2} \\
2.60 \times 10^{-2} & 4.50 \times 10^{-2} & 8.10 \times 10^{-2}
\end{bmatrix}
$$

Inverting the matrix in equation 4, we obtain the stiffness influence coefficient matrix (using Matlab)

$$
K = \begin{bmatrix}
1.40 \times 10^3 & -1.10 \times 10^3 & 2.9 \times 10^2 \\
1.10 \times 10^3 & -3.80 \times 10^2 & 1.70 \times 10^2
\end{bmatrix}
$$

b) Mass discretization: The total mass of the beam is

$$
\rho V = \left(0.1 \text{ lbs/in}^2\right) \left(60'' \times 1'' \times 1''\right) / \left(32.2 \text{ lbs.in}^2 / \text{in}^4\right)
$$
\[ M = 0.016 \text{ slugs} \]

Since we discretized the beam as three masses spaced \( \frac{L}{3} \) from each other, the equivalent mass, \( m_i \), of each mass should be:

\[ m = m_1 = m_2 = m_3 = \frac{M}{3} = 0.0053 \text{ slugs} \]

In matrix form,

\[
M = \begin{bmatrix}
0.0053 & 0 & 0 \\
0 & 0.0053 & 0 \\
0 & 0 & 0.0053
\end{bmatrix} \text{ slugs}
\]

(1) Governing equation and the natural frequencies and modes:

\[
\ddot{q} + \text{equivalent stiffness (from stiffness influence coefficient matrix)} \cdot q = 0
\]

\[
M \ddot{q} + k q = 0
\]

\[
\text{mass matrix } M \quad \text{q is deflection (thus } \ddot{q} \text{ is acceleration)}
\]

The mass matrix is given in equation (I), the stiffness matrix in equation (II) and the deflection vector is

\[
q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}
\]

The natural frequencies can be found by solving equation (II) for the
eigenvalues.

\[
\begin{vmatrix}
1900 - \omega^2(0.0053) & -1100 & 290 \\
-1100 & 1900 - \omega^2(0.0053) & -380 \\
290 & -380 & 190 - \omega^2(0.0053)
\end{vmatrix} = 0
\]

Solving using Matlab, we get:

\[
\begin{align*}
\omega_1 &= 40.9 \text{ rad/s} \\
\omega_2 &= 270 \text{ rad/s} \\
\omega_3 &= 720 \text{ rad/s}
\end{align*}
\]

* Check units of \( k \) and \( \omega \): \( [k] = \left[ \frac{\text{lb}}{\text{in}} \right] = \left[ \frac{\text{slug} \cdot \text{ft}}{\text{s}^2} \right] \left[ \frac{\text{in}}{\text{s}^2} \right] = \left[ \frac{\text{slug} \cdot \text{ft}}{\text{s}^2} \right] \)

\( [\omega] = \left[ \frac{\text{rad}}{\text{s}} \right] \)

* Note: When we solve for the frequency, \( \omega_r \), we get the positive and negative values of the same number, e.g., \( \omega = \pm 40.9 \text{ rad/s} \). However, negative frequency is not physically possible, so it is discarded.

To find the eigenvector associated with each eigenvalue, we substitute in \( \omega_r \) and solve for the vector \( \mathbf{A} \).

\[
[k - \omega^2 \mathbf{m}] \mathbf{A} = 0
\]
For \( \omega_r = \omega_1 = 40.9 \text{ rad/s} \),

\[
[k_e - \omega_m^2] A = \begin{bmatrix}
1900 & -1100 & 290 \\
1050 & -380 & \\
160 & & \\
\end{bmatrix}
\begin{bmatrix}
A_1/A_3 \\
A_2/A_3 \\
1 \\
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
1900 & -1100 \\
-1100 & 1050 \\
\end{bmatrix}
\begin{bmatrix}
A_1/A_3 \\
A_2/A_3 \\
\end{bmatrix} = \begin{bmatrix}
-290 \\
380 \\
\end{bmatrix}
\]

\[
A_1/A_3 = 0.150 \\
A_2/A_3 = 0.51
\]

For \( \omega_r = \omega_2 = 270 \text{ rad/s} \),

\[
[k_e - \omega_m^2] A = \begin{bmatrix}
1500 & -1100 & 290 \\
610 & -380 & \\
-220 & & \\
\end{bmatrix}
\begin{bmatrix}
A_1/A_2 \\
A_2/A_3 \\
1 \\
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
1500 & -1100 \\
-1100 & 610 \\
\end{bmatrix}
\begin{bmatrix}
A_1/A_3 \\
A_2/A_3 \\
\end{bmatrix} = \begin{bmatrix}
-290 \\
380 \\
\end{bmatrix}
\]

\[
A_1/A_3 = -1.1 \\
A_2/A_3 = -1.2
\]
For \( \omega_1 = \omega_3 = 120 \text{ rad/s} \)

\[
[h-j\omega m] = \begin{bmatrix}
-830 & -1100 & 290 \\
-1100 & -380 & -2600 \\
 541 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
A_1/A_2 \\
A_2/A_3 \\
1
\end{bmatrix} = 0
\]

\[
\Rightarrow \begin{bmatrix}
-830 & -1100 \\
-1100 & -1900
\end{bmatrix}
\begin{bmatrix}
A_1/A_3 \\
A_2/A_3
\end{bmatrix} = \begin{bmatrix}
-290 \\
380
\end{bmatrix}
\]

\[\Rightarrow A_1/A_3 = 4.5, \quad A_2/A_3 = -3.2\]

Thus, the eigenvalues and their associated mode shapes can be summarized as follows.

\[
\begin{align*}
\omega_1 &= 40.9 \text{ rad/s} & \varphi_1 &= \begin{bmatrix}
0.150 \\
0.51 \\
1
\end{bmatrix} \\
\omega_2 &= 290 \text{ rad/s} & \varphi_2 &= \begin{bmatrix}
0.11 \\
-1.2 \\
1
\end{bmatrix} \\
\omega_3 &= 120 \text{ rad/s} & \varphi_3 &= \begin{bmatrix}
4.5 \\
-3.2 \\
1
\end{bmatrix}
\end{align*}
\]
3. \[ F(t) = (5.0 \text{lbs}) \sin 2t \]

Sinusoidal force applied at center of simply-supported beam.

a) For a continuous beam, the governing equation is:

\[ EI \frac{d^4 w}{dx^4} + m \frac{d^2 w}{dx^2} = p(t) \]

The general solution for this differential equation is:

\[ w(x, t) = \tilde{w}(x) e^{i \omega t} \]

\[ = [C_1 \sinh lx + C_2 \cosh lx + C_3 \sin lx + C_4 \cos lx] e^{i \omega t} \]

This equation needs 4 boundary conditions for \( C_1 \sim C_4 \), which are

\[ w(0) = 0, \quad w(l) = 0, \quad H = EI \frac{d^2 w}{dx^2} = 0 \]

Plugging equation 2 into equation 3, we get four equations. If we express these equations in matrix form, we obtain:

\[ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ \sinh ll \cosh ll & \cosh ll & -\sinh ll \cos ll & \cos ll \\ \sinh ll \cosh ll & -\sinh ll \cos ll & -\cos ll & \sin ll \cosh ll \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \]
The determinant of the matrix in equation 4 is \( C_3 \sin \lambda l \), and setting this to zero, to get non-trivial solutions, we get

\[ C_3 \sin \lambda l = 0 \]

\[ \Rightarrow \lambda l = n \pi \]

\[ \Rightarrow \frac{m \omega^2}{EI} = \frac{n^2 \pi^2}{l} \]

\[ \Rightarrow \omega = n^2 \pi \sqrt{\frac{EI}{ml}} \]

\[ \omega_r = r^2 \pi \sqrt{\frac{EI}{ml}} \]  \( \text{Eqn. 5} \)

As for multi-mass systems, the associated modes (or eigenvectors), are found by plugging the frequency, \( \omega_r \), back into the matrix governing equation. We find that

\[ \phi_r = \sin \frac{r \pi x}{l} \text{ for } r = 1, 2, 3, \ldots \]  \( \text{Eqn. 6} \)

From equations 5 and 6, the first three frequencies and modes are

\[ \omega_1 = \pi \sqrt{\frac{EI}{ml}} \quad \phi_1 = \sin \frac{\pi x}{l} \]

\[ \omega_2 = 4 \pi \sqrt{\frac{EI}{ml}} \quad \phi_2 = \sin \frac{2 \pi x}{l} \]

\[ \omega_3 = 9 \pi \sqrt{\frac{EI}{ml}} \quad \phi_3 = \sin \frac{3 \pi x}{l} \]
The given data for the beam is: \( E = 10.0 \, \text{ksi} \), \( l = 36'' \),

\[
I = \frac{1}{12} (0.8'')(0.4'')^3 = 4.27 \times 10^{-3} \, \text{in}^4
\]

\[
M = \frac{F}{g} \cdot \frac{\text{volume}}{\text{length}}
\]

\[
= \frac{0.1 \, \text{ft}}{32.2 \, \text{in} \cdot \text{s}^2 \cdot \text{in} / \text{ft}} \cdot (0.8') (0.4')(36')(36'') = 0.1 \, \text{ft} \cdot \text{slug} / \text{in}
\]

\[
\approx 8.29 \times 10^{-3} \, \text{slugs/in}
\]

\# Note: \( M = \frac{\text{mass}}{\text{length}} \) according to definition in unit #23 p.2.

\[
\therefore \sqrt{\frac{EI}{M}} \text{ has units of } \sqrt{\text{slug} \cdot \text{in} / \text{ft} \cdot \text{in}^3} = \frac{1}{5}
\]

Plugging these values into the frequencies and mode shapes, we get

| \( \omega_1 \) | 173 rad/s | \( \phi_1 = \sin \frac{\pi x}{36} \) |
| \( \omega_2 \) | 691 rad/s | \( \phi_2 = \sin \frac{\pi x}{18} \) |
| \( \omega_3 \) | 1560 rad/s | \( \phi_3 = \sin \frac{\pi x}{12} \) |

b) The normal equations of motion have the form

\[
M_r \ddot{\xi}_r + H_r \omega^2 \xi_r = \Xi_r
\]

\# unit #23 p.14

where

\[
M_r = \int_0^l \omega \phi_r^2 \, dx, \quad \Xi_r = \int_0^l \phi_r \phi_0 (x,t) \, dx
\]

\[
\uparrow \quad \text{generalized mass of} \quad \text{rth mode}
\]

\[
\uparrow \quad \text{generalized force of} \quad \text{rth mode}
\]
The applied load for this problem is

\[ P_x(x,t) = F(t) \delta(x - 18\text{"}) \]

Therefore, the generalized force becomes

\[ \sum_r = \int_0^L \varphi_r F(t) \delta(x - 18\text{"}) \, dx \]

\[ = \varphi_r(18\text{"}) F(t) \]

The generalized mass matrix can be evaluated as follows.

\[ M_1 = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \int_0^{36\text{°}} \sin^2 \frac{\pi x}{26} \, dx \]

\[ = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \left[ \frac{1}{2} \frac{\pi}{36} - \frac{1}{4} \sin \frac{\pi x}{18} \right]_0^{36} \]

\[ = 1.30 \times 10^{-4} \text{ slugs} \]

\[ M_2 = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \int_0^{36\text{°}} \sin \frac{\pi x}{18} \, dx \]

\[ = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \left[ \frac{1}{2} \frac{\pi}{18} - \frac{1}{4} \sin \frac{\pi x}{9} \right]_0^{36} \]

\[ = 2.60 \times 10^{-4} \text{ slugs} \]

\[ M_3 = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \int_0^L \sin \frac{\pi x}{12} \, dx \]

\[ = (8.29 \times 10^{-5} \text{ slugs}/\text{in}) \left[ \frac{1}{2} \frac{\pi}{12} - \frac{1}{4} \sin \frac{\pi x}{6} \right]_0^{36} \]

\[ = 3.91 \times 10^{-4} \text{ slugs} \]
The generalized force is

$$\mathbf{E}_1 = \sin \frac{\pi(18^\circ)}{36^\circ} \cdot F(t) = F(t)$$

$$\mathbf{E}_2 = \sin \frac{\pi(18^\circ)}{18^\circ} \cdot F(t) = 0$$

$$\mathbf{E}_3 = \sin \frac{\pi(18^\circ)}{12^\circ} \cdot F(t) = -F(t)$$

Thus, the normal equations of motion for the first three modes are:

\[
\begin{align*}
0.000130 \dddot{\xi}_1 + 8.88 \dddot{\xi}_1 &= F(t) \\
0.000260 \dddot{\xi}_2 + 124 \dddot{\xi}_2 &= 0 \\
0.000291 \dddot{\xi}_3 + 947 \dddot{\xi}_3 &= -F(t)
\end{align*}
\]

d) From b), we know that the general form of the governing decoupled equation is

$$M_r \dddot{\xi}_r + U_r \omega_r^2 \dddot{\xi}_r = \phi_r(18^\circ)(5.0 \text{ lbs}) \sin \omega t$$

The particular solution to this differential equation was obtained in unit #20, p.22.

$$\xi_r = \frac{\phi_r(18^\circ)(5.0 \text{ lbs})}{M_r \omega_r^2 [1 - (\frac{\omega}{\omega_r})^2]} \sin \omega t$$
For \( r=1, 2, 3 \), \( \xi_r \) is

\[
\xi_1 = \frac{1}{(3.885)[1-\left(\frac{c}{\omega_1}\right)^2]} \cdot \frac{1.29}{[1-\left(\frac{c}{\omega_1}\right)^2]} \sin \omega_2 t \quad (\text{in})
\]

\[
\xi_2 = \frac{0}{(12+11.5)[1-\left(\frac{c}{\omega_2}\right)^2]} = 0
\]

\[
\xi_3 = \frac{-1}{(949.4)[1-\left(\frac{c}{\omega_3}\right)^2]} \sin \omega_3 t = -0.00528 \sin \omega_3 t \quad (\text{in})
\]

The deflection at the center is

\[
\omega \left( \frac{L}{2}, t \right) = \sum_{r=1}^{3} \frac{\phi_r \left( \frac{L}{2} \right) \xi_r(t)}{\alpha_r} = \phi_1 \left( \frac{L}{2} \right) s_1(t) + \phi_2 \left( \frac{L}{2} \right) s_2(t) + \phi_3 \left( \frac{L}{2} \right) s_3(t)
\]

\[
= \sin \omega_2 t \left[ \frac{1.29}{[1-\left(\frac{c}{\omega_1}\right)^2]} + \frac{0.00528}{[1-\left(\frac{c}{\omega_3}\right)^2]} \right]
\]

\[
\approx \sin \omega_2 t \left[ \frac{1.29}{[1-\left(\frac{c}{\omega_1}\right)^2]} + \frac{0.00528}{[1-\left(\frac{c}{\omega_3}\right)^2]} \right]
\]

\[
\omega \left( \frac{L}{2}, t \right) = \sin \omega_2 t \left[ \frac{1.29}{[1-\left(\frac{c}{\omega_1}\right)^2]} + \frac{0.00528}{[1-\left(\frac{c}{\omega_3}\right)^2]} \right]
\]

\[
\text{d) The modal responses are plotted on the next page. The amplitudes of the}
\]

\[
\text{become very large and goes to infinity when the natural frequencies}
\]

\[
\omega, \text{ and } \omega_3 \text{ are approached.}
\]