Practice Problems

6. \[
\begin{pmatrix}
H_1 \\ H_2 \\ H_3
\end{pmatrix} =
\begin{bmatrix}
C_{111} & C_{122} & 2C_{112} \\
C_{211} & C_{222} & 2C_{212} \\
C_{311} & C_{322} & 2C_{312}
\end{bmatrix}
\begin{pmatrix}
M_{11} \\
M_{22} \\
M_{12}
\end{pmatrix}
\]

- The \(H\) subscript must be a free index because it changes with equation. It must also be Latin because it takes values 1, 2, 3.

- The \(M\) subscript must be Greek because they take values of 1, 2.

- The second and third \(C\) subscripts are the same as the \(M\) subscripts (therefore, they must also be Greek).

- The first \(C\) subscript matches the \(H\) subscript (therefore, Latin).

If we make the assumptions that \(M_{xy}\) is symmetric and \(C_{xy}\) is symmetric, then the matrix equation given can be written as

\[
H_{i} = C_{i,j} M_{j}\]
\[
\frac{\partial^2 \varepsilon_{mk}}{\partial y_k \partial y} + \frac{\partial^2 \varepsilon_{mn}}{\partial y_m \partial y} - \frac{\partial^2 \varepsilon_{mk}}{\partial y_k \partial y} - \frac{\partial^2 \varepsilon_{mk}}{\partial y_m \partial y} = 0
\]

All of the 4 indices are free indices and take on values 1, 2, 3. Therefore, there are 81 (3x3x3x3) equations that we could obtain. However, they are not all independent, and in fact, some are trivial. Let's consider the symmetries first.

- \( \varepsilon_{mk} = \varepsilon_{km} \Rightarrow \) if we reverse the subscripts, we don't get a new equation.
- \( \varepsilon_{mk} = \varepsilon_{lm} \)
- \( \varepsilon_{nk} = \varepsilon_{mn} \)
- \( \varepsilon_{mk} = \varepsilon_{km} \)

The trivial cases are as follows.

- \( k = l \) \quad results in \( 0 = 0 \)
- \( m = n \)

*Note: From the trivial cases, it can be inferred that if*
any of the three subscripts have the same value, the equation will be trivial.

These conditions leave only 6 independent equations. Any of the six is acceptable.

\[ n = k = 1, \ m = l = 2 \]

\[ \frac{\delta^2 \varepsilon_{11}}{\delta y_1^2} + \frac{\delta^2 \varepsilon_{22}}{\delta y_2^2} - \frac{\delta^2 \varepsilon_{12}}{\delta y_1 \delta y_2} - \frac{\delta^2 \varepsilon_{13}}{\delta y_1 \delta y_3} = 0 \]

\[ \Rightarrow \frac{\delta^2 \varepsilon_{11}}{\delta y_1^2} + \frac{\delta^2 \varepsilon_{22}}{\delta y_2^2} - 2 \frac{\delta^2 \varepsilon_{12}}{\delta y_1 \delta y_2} = 0 \quad (1) \]

\[ n = k = 1, \ m = 2, l = 3 \]

\[ \frac{\delta^2 \varepsilon_{22}}{\delta y_2^2} + \frac{\delta^2 \varepsilon_{33}}{\delta y_3^2} - \frac{\delta^2 \varepsilon_{13}}{\delta y_1 \delta y_3} - \frac{\delta^2 \varepsilon_{12}}{\delta y_1 \delta y_2} = 0 \quad (2) \]

\[ n = k = 1, \ m = l = 3 \]

\[ \frac{\delta^2 \varepsilon_{11}}{\delta y_1^2} + \frac{\delta^2 \varepsilon_{22}}{\delta y_2^2} - \frac{\delta^2 \varepsilon_{12}}{\delta y_1 \delta y_2} - \frac{\delta^2 \varepsilon_{13}}{\delta y_1 \delta y_3} = 0 \]

\[ \Rightarrow \frac{\delta^2 \varepsilon_{11}}{\delta y_1^2} + \frac{\delta^2 \varepsilon_{22}}{\delta y_2^2} - 2 \frac{\delta^2 \varepsilon_{12}}{\delta y_1 \delta y_2} = 0 \quad (3) \]
\[ n=1, \; k=2, \; m=3, \; l=3 \]

\[ \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} + \frac{\partial^2 \sigma_{zz}}{\partial y_2 \partial y_3} - \frac{\partial^2 \sigma_{zz}}{\partial y_2^2} - \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} = 0 \quad (4) \]

\[ n=1, \; k=2, \; m=\ell=3 \]

\[ \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} + \frac{\partial^2 \sigma_{zz}}{\partial y_2 \partial y_3} - \frac{\partial^2 \sigma_{zz}}{\partial y_2^2} - \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} = 0 \quad (5) \]

\[ n=2, \; k=3, \; m=3, \; \ell=3 \]

\[ \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} + \frac{\partial^2 \sigma_{zz}}{\partial y_2^2} - \frac{\partial^2 \sigma_{zz}}{\partial y_2^2} - \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} = 0 \]

\[ \Rightarrow \frac{\partial^2 \sigma_{zz}}{\partial y_3^2} + \frac{\partial^2 \sigma_{zz}}{\partial y_2^2} - 2 \frac{\partial^2 \sigma_{zz}}{\partial y_2 \partial y_3} = 0 \quad (6) \]

\* Note: Due to the symmetries, other combination of indices \( n, k, m, \ell \) will also give these equations.

To convert to engineering notation, note the following differences.

1. **Index conversion**
   
   \[
   \begin{align*}
   11 & \rightarrow x \\
   12 & \rightarrow xy, \ yz \\
   22 & \rightarrow y \\
   13 & \rightarrow xz, \ 2x \\
   33 & \rightarrow z \\
   23 & \rightarrow yz, \ 2y
   \end{align*}
   \]
② Shear strain conversion

\[ 2 \varepsilon_{mn} = \gamma_{mn} \quad \text{for} \quad m \neq n \]

③ Axis conversion

\[ y_1 \rightarrow x, \quad y_2 \rightarrow y, \quad y_3 \rightarrow z \]

The six equations can be written in engineering notation as follows:

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad (1) \\
\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + 2 \frac{\partial^2 \varepsilon_{yy}}{\partial x \partial y} - 2 \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad (2) \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} - \frac{\partial^2 \gamma_{yz}}{\partial x \partial z} = 0 \quad (3) \\
2 \frac{\partial^2 \gamma_{xy}}{\partial y \partial z} + 2 \frac{\partial^2 \gamma_{yz}}{\partial x \partial y} - 2 \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \quad (4) \\
2 \frac{\partial^2 \gamma_{xy}}{\partial z \partial y} + \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial x} - 2 \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} - 2 \frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} = 0 \quad (5) \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = 0 \quad (6) \]
These equations are often written in the following form.

\[
\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}
\]

\[
\frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2}
\]

\[
\frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2}
\]

2 \frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial}{\partial x} \left( - \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z} \right)

2 \frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z} \right)

2 \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial z} \right)