Norm-Up Exercises

1. To find the expressions for the stress components at the hole boundary as a function of $\theta$, we will use the Airy stress functions. The problem we need to consider is an isotropic plate with a hole under compressive load.

\[ \sigma_\theta = \frac{\pi r}{2} \]

From our notes (unit 8, p.20), our assumed stress function, $\phi$, for an isotropic plate with a hole in polar coordinates is

\[ \phi(r, \theta) = [A_0 + B_0 \ln r + C_0 r^2 + D_0 r^4 \ln r] \\
+ [A_1 r^3 + \frac{B_1}{r} + C_1 r^4 + D_1] \cos 2\theta - 1 \]
In order for the displacements to be single-valued, \( D_0 \) is set to equal to zero. For the stresses to be bounded as \( r \to \infty \), \( C_2 \) also needs to be zero. The stress in polar coordinates can now be expressed as:

\[
\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}
= \frac{B_0}{r^3} + 2C_0 - \left[ 2A_2 + \frac{6B_3}{r^6} \right] \cos \theta \quad (2)
\]

\[
\sigma_{\theta \theta} = \frac{\partial^2 \phi}{\partial r^2}
= -\frac{B_0}{r^3} + 2C_0 + \left[ 2A_2 + \frac{6B_3}{r^6} \right] \cos \theta \quad (3)
\]

\[
\sigma_{r \theta} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta}
= \left[ 2A_2 - \frac{6B_3}{r^6} - \frac{2B_4}{r^8} \right] \sin \theta \quad (4)
\]

The boundary conditions are:

\( \sigma_{rr} = \sigma_{\theta \theta} = 0 \quad (5) \quad r = 0 \) ← stress-free at hole edge.

\( \sigma_{yy} = \sigma_0 , \quad \sigma_{yy} = 0 \quad (6) \quad y_1 \to \infty \)

\( \sigma_{xx} = 0 , \quad \sigma_{xy} = 0 \quad (7) \quad y_1 \to \infty \)
Using these boundary conditions, we can find the unknown constants in equations (2) through (4). Shifting the mode

as described in notes, we can get

\[ \sigma_{rr} = \frac{G_1}{2} \left( 1 - \frac{a^2}{r^2} \right) + \frac{G_0}{2} \left( 1 - 4 \frac{a^2}{r^2} + 3 \frac{a^4}{r^4} \right) \cos 2\theta \] — (5)

\[ \sigma_{\theta\theta} = \frac{G_1}{2} \left( 1 + \frac{a^2}{r^2} \right) - \frac{G_0}{2} \left( 1 + 3 \frac{a^2}{r^2} \right) \cos 2\theta \] — (6)

\[ \sigma_{r\theta} = -\frac{G_1}{2} \left( 1 + 2 \frac{a^2}{r^2} - 3 \frac{a^4}{r^4} \right) \sin 2\theta \] — (7)

At the hole boundary, i.e., \( r = a \), the stresses are reduced to

\[ \sigma_{rr} = 0 \] — (8)

\[ \sigma_{\theta\theta} = -G_0 \left( 1 - 2 \cos 2\theta \right) \] — (9)

\[ \sigma_{r\theta} = 0 \] — (10)

Now, we need to transform our stresses from the polar to Cartesian system.
Set
\[ \tilde{\sigma}_{11} = \tilde{\sigma}_{11} = 0 \]
\[ \tilde{\sigma}_{22} = \tilde{\sigma}_{22} \]
\[ \tilde{\sigma}_{33} = \tilde{\sigma}_{33} = 0 \]

To transform our \( r, \theta \) (i.e. \( \hat{y}_1, \hat{y}_2 \)) system to our \( y_1, y_2 \) system, we must rotate through an angle \(-90 + \theta\).

The transformation rule is
\[ \tilde{\sigma}_{x\beta} = l_{x\sigma} l_{\beta\lambda} \tilde{\sigma}_{\sigma\lambda} \]

This equation reduces to
\[ \sigma_{11} = l_{11} l_{11} \tilde{\sigma}_{11} = \cos^2 \theta \sigma_{11} \]
\[ \sigma_{22} = l_{22} l_{22} \tilde{\sigma}_{22} = \cos^2 (90 + \theta) \sigma_{22} = \sin^2 \theta \sigma_{22} \]
\[ \sigma_{33} = l_{33} l_{33} \tilde{\sigma}_{33} = (-\cos \theta)(-\sin \theta) \sigma_{33} = \sin \theta \cos \theta \sigma_{33} \]

\[ \times \]
\[ l_{11} = \cos(180 - \theta) = -\cos \theta \]
\[ l_{33} = \cos(90 - \theta) = -\sin \theta \]
Therefore, plugging the expression for \( \sigma_{\theta\theta} \) in equations (3) through (8), we get:

\[
\begin{align*}
\sigma_{\theta\theta} &= -\sigma_0 (1 - 2\cos^2 \theta) 
\sigma_{\phi\phi} &= -\sigma_0 (1 - 2\cos^2 \theta) 
\sigma_{\phi\theta} &= \sigma_0 (1 - 2\cos^2 \theta) \sin \theta \cos \theta
\end{align*}
\]

For use in problems 2 and 3, the stress can be expressed in normalized form as:

**\( y_1, y_2 \) coordinate:**

\[
\begin{align*}
\frac{\sigma_{\theta\theta}}{\sigma_0} &= -(1 - 2\cos^2 \theta) 
\frac{\sigma_{\phi\phi}}{\sigma_0} &= -(1 - 2\cos^2 \theta) 
\frac{\sigma_{\phi\theta}}{\sigma_0} &= (1 - 2\cos^2 \theta) \sin \theta \cos \theta
\end{align*}
\]

**\( r, \theta \) coordinate:**

\[
\begin{align*}
\frac{\sigma_{r\theta}}{\sigma_0} &= 0 
\frac{\sigma_{\theta\theta}}{\sigma_0} &= -(1 - 2\cos^2 \theta) 
\frac{\sigma_{\phi\theta}}{\sigma_0} &= 0
\end{align*}
\]
2. Stresses in \( y_1 - y_2 \) coordinate system

![Stresses in \( y_1 - y_2 \) Coordinates](image)

3. Stresses in \( r - \theta \) coordinate system

![Stresses in \( r - \theta \) Coordinates](image)
4. We can derive a number of interesting results from these plots.

1. Extensional stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$) are symmetric about each 90° rotation.

2. Shear stresses ($\sigma_{xy}$) are anti-symmetric about each 90° rotation.

3. All stresses go to zero at $\theta = 30^\circ$, $150^\circ$, $210^\circ$ and $330^\circ$.

4. $\sigma_{xx}$ and $\sigma_{yy}$ are zero at hole edge. Only $\sigma_{zz}$ is non-zero.

5. In the plots, all stresses are normalized by $\sigma_{xx}$, which is negative in the present case. So, the actual stress state has the same slope but opposite sign.