Unit 10
St. Venant Torsion Theory

Readings:
Rivello 8.1, 8.2, 8.4
T & G 101, 104, 105, 106

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III. Torsion
We have looked at basic in-plane loading. Let’s now consider a second “building block” of types of loading: basic torsion.

There are 3 basic types of behavior depending on the type of cross-section:

1. Solid cross-sections

   “classical” solution technique via stress functions

2. Open, thin-walled sections

   Membrane Analogy
3. **Closed, thin-walled sections**

In Unified you developed the basic equations based on some broad assumptions. Let’s…

- Be a bit more rigorous
- Explore the limitations for the various approaches
- Better understand how a structure “resists” torsion and the resulting deformation
- Learn how to model general structures by these three basic approaches

Look first at
Classical (St. Venant’s) Torsion Theory

Consider a long prismatic rod twisted by end torques:
\[ T \text{ [in - lbs]} \quad [m - n] \]

*Figure 10.1 Representation of general long prismatic rod*

Length \((l)\) >> dimensions in x and y directions

Do not consider *how* end torque is applied (St. Venant’s principle)
Assume the following **geometrical behavior:**

a) Each cross-section (@ each z) rotates as a rigid body (No "distortion" of cross-section shape in x, y)

b) Rate of twist, \( k = \text{constant} \)

c) Cross-sections are free to warp in the z-direction but the warping is the same for all cross-sections

  This is the "St. Venant Hypothesis"

"warping" = extensional deformation in the direction of the axis about which the torque is applied

Given these assumptions, we see if we can satisfy the equations of elasticity and B.C.'s.

\[ \Rightarrow \text{SEMI-INVERSE METHOD} \]

Consider the deflections:

Assumptions imply that at any cross-section location \( z \):

\[ \alpha = \left( \frac{d\alpha}{dz} \right) z = k z \]

(careful! Rivello uses \( \phi \)!)

rate of twist

a constant

(define as 0 @ \( z = 0 \))
**Figure 10.2**  Representation of deformation of cross-section due to torsion

This results in:

- \( u(x, y, z) = r\alpha (-\sin \beta) \)
- \( v(x, y, z) = r\alpha (\cos \beta) \)
- \( w(x, y, z) = w(x, y) \)

\( \Rightarrow \) independent of \( z \)!
We can see that:

\[ r = \sqrt{x^2 + y^2} \]

\[ \sin \beta = \frac{y}{r} \]

\[ \cos \beta = \frac{x}{r} \]

This gives:

\[ u(x, y, z) = -y \ k \ z \] \hspace{1cm} (10 - 1) \]

\[ v(x, y, z) = x \ k \ z \] \hspace{1cm} (10 - 2) \]

\[ w(x, y, z) = w(x, y) \] \hspace{1cm} (10 - 3) \]
Next look at the Strain-Displacement equations:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = 0 \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0 \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0
\end{align*}
\]

(consider: \(u\) exists, but \(\frac{\partial u}{\partial x} = 0\)
\(v\) exists, but \(\frac{\partial v}{\partial y} = 0\))

\(\Rightarrow\) No extensional strains in torsion if cross-sections are free to warp
\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z k + z k = 0 \]

\[ \Rightarrow \text{cross-section does not change shape (as assumed!)} \]

\[ \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = k x + \frac{\partial w}{\partial y} \]  \hspace{1cm} (10 - 4)

\[ \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -k y + \frac{\partial w}{\partial x} \]  \hspace{1cm} (10 - 5)

Now the Stress-Strain equations:

let's first do isotropic

\[ \varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz}) \right] = 0 \]

\[ \varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz}) \right] = 0 \]

\[ \varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy}) \right] = 0 \]

\[ \Rightarrow \sigma_{xx}, \sigma_{yy}, \sigma_{zz} = 0 \]
\[ \varepsilon_{xy} = \frac{2(1 + \nu)}{E} \quad \sigma_{xy} = 0 \implies \sigma_{xy} = 0 \]

\[ \varepsilon_{yz} = \frac{2(1 + \nu)}{E} \sigma_{yz} \quad (10 - 6) \]

\[ \varepsilon_{xz} = \frac{2(1 + \nu)}{E} \sigma_{xz} \quad (10 - 7) \]

\[ \implies \text{only } \sigma_{xz} \text{ and } \sigma_{yz} \text{ stresses exist} \]

Look at **orthotropic** case:

\[ \varepsilon_{xx} = \frac{1}{E_{11}} \left[ \sigma_{xx} - \nu_{12} \sigma_{yy} - \nu_{13} \sigma_{zz} \right] = 0 \]

\[ \varepsilon_{yy} = \frac{1}{E_{22}} \left[ \sigma_{yy} - \nu_{21} \sigma_{xx} - \nu_{23} \sigma_{zz} \right] = 0 \]

\[ \varepsilon_{zz} = \frac{1}{E_{33}} \left[ \sigma_{zz} - \nu_{31} \sigma_{xx} - \nu_{32} \sigma_{yy} \right] = 0 \]

\[ \implies \sigma_{xx}, \sigma_{yy}, \sigma_{zz} = 0 \text{ still equal zero} \]
\[ \varepsilon_{yz} = \frac{1}{G_{23}} \sigma_{yz} \]

\[ \varepsilon_{xz} = \frac{1}{G_{13}} \sigma_{xz} \]

Differences are in \( \varepsilon_{yz} \) and \( \varepsilon_{xz} \) here as there are two different shear moduli \((G_{23} \text{ and } G_{13})\) which enter in here.

For \textit{anisotropic material}:

coefficients of mutual influence and Chentsov coefficients foul everything up (no longer "simple" torsion theory). [can’t separate torsion from extension]

Back to general case...

Look at the \textit{Equilibrium Equations}:

\[ \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad \Rightarrow \quad \sigma_{xz} = \sigma_{xz} (x, y) \]

\[ \frac{\partial \sigma_{yz}}{\partial z} = 0 \quad \Rightarrow \quad \sigma_{yz} = \sigma_{yz} (x, y) \]
So, $\sigma_{xz}$ and $\sigma_{yz}$ are only functions of $x$ and $y$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \quad (10 - 8)$$

We satisfy equation (10 - 8) by introducing a Torsion (Prandtl) Stress Function $\phi(x, y)$ where:

$$\frac{\partial \phi}{\partial y} = -\sigma_{xz} \quad (10 - 9a)$$

$$\frac{\partial \phi}{\partial x} = \sigma_{yz} \quad (10 - 9b)$$

Using these in equation (10 - 8) gives:

$$\frac{\partial}{\partial x} \left( -\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial x} \right) \equiv 0$$

$\Rightarrow$ Automatically satisfies equilibrium (as a stress function is supposed to do)
Now consider the **Boundary Conditions**:

(a) Along the contour of the cross-section

*Figure 10.3*  **Representation of stress state along edge of solid cross-section under torsion**

*Figure 10.4*  **Close-up view of edge element from Figure 10.3**

- $\sigma_{xz}$ (into page)
- $\sigma_{yz}$ (out of page)
Using equilibrium:
\[ \sum F_z = 0 \] (out of page is positive)
gives:
\[-\sigma_{xz} \, dydz + \sigma_{yz} \, dxdz = 0\]

Using equation (10 - 9) results in
\[ - \left( - \frac{\partial \phi}{\partial y} \, dy \right) + \left( \frac{\partial \phi}{\partial x} \right) dx = 0 \]
\[ \left( \frac{\partial \phi}{\partial y} \, dy \right) + \left( \frac{\partial \phi}{\partial x} \, dx \right) = d\phi \]

And this means:
\[ d\phi = 0 \]
\[ \Rightarrow \phi = \text{constant} \]

We take:
\[ \boxed{\phi = 0} \] along contour (10 - 10)

Note: addition of an arbitrary constant does not affect the stresses, so choose a convenient one (0!)
Boundary condition (b) on edge \( z = 1 \)

**Figure 10.5** Representation of stress state at top cross-section of rod under torsion

![Figure 10.5](image)

Equilibrium tells us the force in each direction:

\[
F_x = \int \int \sigma_{zx} \, dx \, dy
\]

using equation (10 - 9):

\[
= \int \int y_R \frac{\partial \phi}{\partial y} \, dx \, dy
\]

where \( y_R \) and \( y_L \) are the geometrical limits of the cross-section in the \( y \) direction.
\[= - \int_{y_L}^{y_R} [\phi]_y^x \, dx\]

and since \(\phi = 0\) on contour

\[F_x = 0 \quad \text{O.K.} \quad \text{(since no force is applied in x-direction)}\]

Similarly:

\[F_y = \int \int \sigma_{zy} \, dxdy = 0 \quad \text{O.K.}\]

Look at one more case via equilibrium:

Torque \(= T = \int \int [x\sigma_{zy} - y\sigma_{zx}] \, dxdy\)

\[= \int \int_{x_T}^{x_B} x \frac{\partial \phi}{\partial x} \, dxdy + \int \int_{y_L}^{y_R} y \frac{\partial \phi}{\partial y} \, dydx\]

where \(x_T\) and \(x_B\) are geometrical limits of the cross-section in the x-direction

Integrate each term by parts:

\[\int AdB = AB - \int BdA\]
Set:
\[ A = x \Rightarrow dA = dx \]
\[ dB = \frac{\partial \phi}{\partial x} dx \Rightarrow B = \phi \]
and similarly for \( y \)
\[ T = \int \left[ x\phi \right]_x^X \frac{dX}{dx} dy + \int \left[ y\phi \right]_y^Y \frac{dY}{dy} dx \]
\[ = 0 \quad \text{since } \phi = 0 \text{ in contour} \]
\[ = 0 \quad \text{since } \phi = 0 \text{ in contour} \]
\[ \Rightarrow T = -2 \int \int \phi \, dx \, dy \quad (10 - 11) \]

Up to this point, all the equations [with the slight difference in stress-strain of equations (10 - 6) and (10 - 7)] are also valid for orthotropic materials.
Summarizing

- Long, prismatic bar under torsion
- Rate of twist, $k = \text{constant}$
- $\varepsilon_{yz} = kx + \frac{\partial w}{\partial y}$
- $\varepsilon_{xz} = -ky + \frac{\partial w}{\partial x}$
  \[ \frac{\partial \phi}{\partial y} = -\sigma_{xz} \quad \frac{\partial \phi}{\partial x} = \sigma_{yz} \]

- Boundary conditions
  \[ \phi = 0 \text{ on contour (free boundary)} \]
  \[ T = -2 \iint \phi \, dx\,dy \]
Solution of Equations

(now let’s go back to isotropic)

Place equations (10 - 4) and (10 - 5) into equations (10 - 6) and (10 - 7) to get:

\[
\sigma_{yz} = G\varepsilon_{yz} = G \left( k x + \frac{\partial w}{\partial y} \right) \quad (10 - 12)
\]

\[
\sigma_{xz} = G\varepsilon_{xz} = G \left( - k y + \frac{\partial w}{\partial x} \right) \quad (10 - 13)
\]

We want to eliminate \( w \). We do this via:

\[
\frac{\partial}{\partial x} \{\text{Eq. (10 - 12)}\} - \frac{\partial}{\partial y} \{\text{Eq. (10 - 13)}\}
\]

to get:

\[
\frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y} = G \left( k + \frac{\partial^2 w}{\partial x \partial y} + k - \frac{\partial^2 w}{\partial y \partial x} \right)
\]
and using the definition of the stress function of equation (10 - 9) we get:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2Gk \]  \hspace{1cm} (10 - 14)

Poisson's Equation for \( \phi \)
(Nonhomogeneous Laplace Equation)

**Note for orthotropic material**

We do **not** have a common shear modulus, so we would get:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (G_{xz} + G_{yz}) k + \left( G_{yz} - G_{xz} \right) \frac{\partial^2 w}{\partial x \partial y} \]

⇒ We cannot eliminate \( w \) unless \( G_{xz} \) and \( G_{yz} \) are virtually the same.
Overall solution procedure:

- Solve Poisson equation (10 - 14) subject to the boundary condition of $\phi = 0$ on the contour
- Get $T - k$ relation from equation (10 - 11)
- Get stresses $(\sigma_{xz}, \sigma_{yz})$ from equation (10 - 9)
- Get $w$ from equations (10 - 12) and (10 - 13)
- Get $u, v$ from equations (10 - 1) and (10 - 2)
- Can also get $\varepsilon_{xz}, \varepsilon_{yz}$ from equations (10 - 6) and (10 - 7)

This is “St. Venant Theory of Torsion”

Application to a Circular Rod

Figure 10.6  Representation of circular rod under torsion cross-section
“Let”:

\[ \phi = C_1 (x^2 + y^2 - R^2) \]

This satisfies \( \phi = 0 \) on contour since \( x^2 + y^2 = R^2 \) on contour

This gives:

\[ \frac{\partial^2 \phi}{\partial x^2} = 2C_1 \quad \frac{\partial^2 \phi}{\partial y^2} = 2C_1 \]

Place these into equation (10-14):

\[ 2C_1 + 2C_1 = 2Gk \]

\[ \Rightarrow C_1 = \frac{Gk}{2} \]

Note: (10-14) is satisfied exactly
Thus:

$$\phi = \frac{Gk}{2} \left( x^2 + y^2 - R^2 \right)$$

Satisfies boundary conditions and partial differential equation exactly

Now place this into equation (10-11):

$$T = -2 \int\int \phi \, dx\,dy$$

**Figure 10.7** Representation of integration strip for circular cross-section

$$T = Gk \int_{-R}^{R} \int_{-\sqrt{R^2 - y^2}}^{+\sqrt{R^2 - y^2}} (R^2 - y^2 - x^2) \, dx\,dy$$
\[ T = Gk \left[ \int_{-R}^{R} \left( R^2 - y^2 \right) x - \frac{x^3}{3} \right] \frac{1}{\sqrt{R^2 - y^2}} dy \]

\[ = Gk \left[ \frac{4}{3} \int_{-R}^{R} (R^2 - y^2)^{3/2} dy \right] \]

\[ = Gk \left[ \frac{4}{3} \frac{1}{4} \left[ y(R^2 - y^2)^{3/2} + \frac{3}{2} R^2 y \sqrt{R^2 - y^2} + \frac{3}{2} R^4 \sin^{-1} \frac{y}{R} \right] \right] \]

\[ = 0 \quad = 0 \quad = \frac{3}{2} R^4 \pi \]

This finally results in

\[ T = Gk \frac{\pi R^4}{2} \]
Since $k$ is the rate of twist: $k = \frac{d\alpha}{dz}$, we can rewrite this as:

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

where:

- $J = \text{torsion constant}$
  
  $J = \frac{\pi R^4}{2}$ for a circle

- $\alpha = \text{amount of twist}$

and:

$GJ = \text{torsional rigidity}$

**Note** similarity to:

$$\frac{d^2 w}{dx^2} = \frac{M}{EI}$$

where: $EI = \text{bending rigidity}$

- (I) $J - \text{geometric part}$
- (E) $G - \text{material part}$
To get the stresses, use equation (10 - 9):

\[
\sigma_{yz} = \frac{\partial\phi}{\partial x} = Gkx = \frac{T}{J} x
\]

\[
\sigma_{xz} = -\frac{\partial\phi}{\partial y} = -Gky = -\frac{T}{J} y
\]

**Figure 10.8** Representation of resultant shear stress, \(\tau_{\text{res}}\), as defined

**Define** a resultant stress:

\[
\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}
\]

\[
= \frac{T}{J} \sqrt{x^2 + y^2}
\]

\[= r\]
The final result is:

\[ \tau = \frac{Tr}{J} \]

for a circle

**Note:** similarity to \( \sigma_x = -\frac{M_z}{I} \)

\( \tau \) always acts along the contour (shape) resultant

**Figure 10.9** Representation of shear resultant stress for circular cross-section

No shear stress on surface
Also note:

1. Contours of $\phi$: close together near edge $\Rightarrow$ higher $\tau$

*Figure 10.10* Representation of contours of torsional shear function

2. Stress pattern ($\tau$) creates twisting

*Figure 10.11* Representation of shear stresses acting perpendicular to radial lines
To get the deflections, first find $\alpha$:

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

(pure rotation of cross-section)

Integration yields:

$$\alpha = \frac{Tz}{GJ} + C_1$$

Let $C_1 = 0$ by saying $\alpha = 0 \text{ @ } z = 0$

Use equations (10 - 1) and (10 - 2) to get:

$$u = -yzk = -y \frac{Tz}{GJ}$$

$$v = xzk = x \frac{Tz}{GJ}$$

Go to equations (10 - 12) and (10 - 13) to find $w(x, y)$:

Equation (10 - 12) gives:

$$\frac{\partial w}{\partial y} = \frac{\sigma_{yz}}{G} - kx$$
using the result for $\sigma_{yz}$:

$$\frac{\partial w}{\partial y} = \frac{Gkx}{G} - kx = 0$$

integration of this says

$$w(x, y) = g_1(x) \quad (not \ a \ function \ of \ y)$$

In a similar manner…

Equation (10 -13) gives:

$$\frac{\partial w}{\partial x} = \frac{\sigma_{xz}}{G} + ky$$

Using $\sigma_{xz} = -Gky$ gives:

$$\frac{\partial w}{\partial x} = -\frac{Gky}{G} + ky = 0$$

integration tells us that:

$$w(x, y) = g_2(y) \quad (not \ a \ function \ of \ x)$$

Using these two results we see that if $w(x, y)$ is neither a function of $x$ nor $y$, then it must be a constant. Might as well take this as \textbf{zero}.
(other constants just show a rigid displacement in $z$ which is trivial)

$\Rightarrow w(x, y) = 0$  \textbf{No warping for circular cross-sections}

(this is the only cross-section that has no warping)

\textbf{Other Cross-Sections}

In other cross-sections, warping is “the ability of the cross-section to resist torsion by differential bending”.

2 parts for torsional rigidity

- Rotation
- Warping

\textbf{Ellipse}
\[ \phi = C_1 \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \]

**Equilateral Triangle**

\[ \phi = C_1 \left( x - \sqrt{3}y + \frac{2}{3}a \right) \left( x + \sqrt{3}y - \frac{2}{3}a \right) \left( x + \frac{1}{3}a \right) \]

**Rectangle**
\[ \phi = \sum_{n \text{ odd}} \left( C_n + D_n \cosh \frac{n\pi y}{b} \right) \cos \frac{n\pi x}{a} \]

Series: (the more terms you take, the better the solution)

These all give solutions to \( \nabla^2 \phi = 2GK \) subject to \( \phi = 0 \) on the boundary. In general, there \textbf{will} be warping

\textit{see Timoshenko for other relations (Ch. 11)}

\textbf{Note:} there are also solutions via “warping functions”. This is a displacement formulation

\textit{see Rivello 8.4}

Next we’ll look at an analogy used to “solve” the general torsion problem