Unit 17
The Beam-Column

Readings:
Theory of Elastic Stability, Timoshenko (and Gere), McGraw-Hill, 1961 (2nd edition), Ch. 1

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Thus far have considered separately:

- beam -- takes bending loads
- column -- takes axial loads

Now combine the two and look at the “beam-column”

(Note: same geometrical restrictions as on others: \( l >> \) cross-sectional dimensions)

Consider a beam with an axial load (general case):

Figure 17.1  Representation of beam-column

(could also have \( p_y \) for bending in y direction)

Consider 2-D case:
Cut out a deformed element $dx$:

**Figure 17.2** Loads and moment acting on deformed infinitesimal element of beam-column

Assume small angles such that:

$$\sin \frac{dw}{dx} \approx \frac{dw}{dx}$$

$$\cos \frac{dw}{dx} \approx 1$$
Sum forces and moments:

\[ \sum F_x = 0 \quad \Rightarrow \quad : \]

\[- F + F + \frac{dF}{dx} dx + p_x dx \]

\[- S \frac{dw}{dx} + \left( S + \frac{dS}{dx} \right) \left( \frac{dw}{dx} + \frac{d^2w}{dx^2} dx \right) = 0 \]

This leaves:

\[ \frac{dF}{dx} dx + p_x dx + \left( \frac{dS dw}{dx dx} + S \frac{d^2w}{dx^2} \right) dx + \text{H.O.T.} = 0 \]

\[ \Rightarrow \quad \frac{dF}{dx} = -p_x - \frac{d}{dx} \left( S \frac{dw}{dx} \right) \quad (17-1) \]
\[ \sum F_z = 0 \]

\[ -F \frac{dw}{dx} + \left( F + \frac{dF}{dx} \right) \left( \frac{dw}{dx} + \frac{d^2w}{dx^2} dx \right) + S - \left( S + \frac{dS}{dx} \right) + p_z dx = 0 \]

This results in:

\[ \frac{dS}{dx} = p_z + \frac{d}{dx} \left( F \frac{dw}{dx} \right) \quad (17-2) \]

new term

\[ \sum M_y = 0 \]

\[ -M + M + \frac{dM}{dx} dx + p_z dx \frac{dx}{2} - p_x dx \frac{dw}{dx} \frac{dx}{2} - \left( S + \frac{dS}{dx} \right) dx = 0 \]

(Using the previous equations) this results in:
\[ \frac{dM}{dx} = S \]  \hspace{1cm} (17-3)

Note: same as before (for Simple Beam Theory)

Recall from beam bending theory:

\[ M = EI \frac{d^2w}{dx^2} \]  \hspace{1cm} (17-4)

Do some manipulating - place (17-4) into (17-3):

\[ S = \frac{d}{dx} \left( EI \frac{d^2w}{dx^2} \right) \]  \hspace{1cm} (17-5)

and place this into (17-2) to get:

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) - \frac{d}{dx} \left( F \frac{dw}{dx} \right) = p_z \]  \hspace{1cm} (17-6)

Basic differential equation for Beam-Column --
(Bending equation -- fourth order differential equation)
To find the axial force $F(x)$, place (17-5) into (17-1):

$$\frac{dF}{dx} = -p_x - \frac{d}{dx} \left[ \frac{dw}{dx} \frac{d}{dx} \left( EI \frac{d^2w}{dx^2} \right) \right]$$

For $w$ small, this latter part is a second order term in $w$ and is therefore negligible.

Thus:

$$\frac{dF}{dx} = -p_x \quad (17-7)$$

**Note:** Solve this equation first to find $F(x)$ distribution and use that in equation (17-6)

**Examples** of solution to Equation (17-7)

- End compression $P_o$

*Figure 17.3*  Simply-supported column under end compression

$$p_x = 0$$
\[
\frac{dF}{dx} = 0 \implies F = C_1
\]

Find \(C_1\) via boundary condition \(\@ x = 0, \ F = -P_0 = C_1\)
\[
\implies F = -P_0
\]

- Beam under its own weight

**Figure 17.4** Representation of end-fixed column under its own weight

\[p_x = -mg\]
\[
\frac{dF}{dx} = +mg \implies F = mgx + C_1
\]

Boundary condition: \(\@ x = \ell, \ F = 0\)

So: \(mg\ell + C_1 = 0 \implies C_1 = -mg\ell\)
\[ F = -mg (\ell - x) \]

- Helicopter blade

*Figure 17.5 Representation of helicopter blade*

(radial force due to rotation)

similar to previous case

Once have \( F(x) \), proceed to solve equation (17-6). Since it is fourth order, need four boundary conditions (two at each end of the beam-column)

-- same possible boundary conditions as previously enumerated

Notes:

- When \( EI \to 0 \), equation (17-6) reduces to:

\[ -\frac{d}{dx} \left( F \frac{dw}{dx} \right) = p_z \]

this is a **string** (second order \( \Rightarrow \) only need two boundary conditions -- one at each end)
(also note that a string cannot be clamped since it cannot carry a moment)

- If \( F = 0 \), get:
  \[
  \frac{d^2}{dx^2} \left( EI \frac{d^2 w}{dx^2} \right) = p_z
  \]
  and for \( EI \) constant:
  \[
  EI \frac{d^4 w}{dx^4} = p_z \quad \text{(basic bending equation)}
  \]

- For \( p_z = 0 \), \( EI \) constant, and \( F \) constant (\( = -P \)), get:
  \[
  EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0 \quad \text{(basic buckling equation)}
  \]

**Buckling of Beam-Column**

Consider the overall geometry (assume beam-column initially straight)
Figure 17.6  Representation of general configuration of beam-column

Cut the beam-column:

Figure 17.7  Representation of beam-column with cut to determine stress resultants

\[ \sum M = 0 : \quad M - M_{\text{primary}} + Pw = 0 \]

due to transverse loading  \quad secondary moment (due to deflection)
gives:

\[ M = EI \frac{d^2w}{dx^2} = M_{\text{primary}} - Pw \]

for transverse loading:

\[ \frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) - \frac{d}{dx} \left( F \frac{dw}{dx} \right) = p_z \]

integrate twice with \( F = -P = C_1 \)

\[ EI \frac{d^2w}{dx^2} + Pw = M_{\text{primary}} \]

same equation as by doing equilibrium

Solve this by:

- getting homogenous solution for \( w \)
- getting particular solution for \( M_{\text{primary}} \)
- applying boundary condition
Figure 17.8  Representation of moment(s) versus applied load for beam-column

Examples
- “Old” airplanes w/struts
- Space structure undergoing rotation

**Final note:** The beam-column is an important concept and the moments in a beam-column can be much worse/higher than beam theory or a perfect column alone.