Unit 3
(Review of) Language of Stress/Strain Analysis

Readings:
B, M, P  A.2, A.3, A.6
Rivello  2.1, 2.2
T & G    Ch. 1 (especially 1.7)

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Professor of Aeronautics & Astronautics
and Engineering Systems
Recall the definition of stress:

\[ \sigma = \text{stress} = \text{“intensity of internal force at a point”} \]

**Figure 3.1** Representation of cross-section of a general body

There are two types of stress:

1. Normal (or extensional): act normal to the plane of the element
2. Shear: act in-plane of element

Sometimes delineated as \( \tau \)
And recall the definition of \textbf{strain}:

\[ \varepsilon = \text{strain} = \text{“percentage deformation of an infinitesimal element”} \]

\textit{Figure 3.2} \hspace{1em} \textbf{Representation of 1-Dimensional Extension of a body}

\[ \varepsilon = \lim_{L \to 0} \left( \frac{\Delta L}{L} \right) \]

Again, there are two types of strain:

\begin{itemize}
  \item \( \varepsilon_n \): Normal (or extensional): elongation of element
  \item \( \varepsilon_s \): Shear: angular change of element
\end{itemize}

\[ \rightarrow \hspace{1em} \text{Sometimes delineated as } \gamma \]

\textit{Figure 3.3} \hspace{1em} \textbf{Illustration of Shear Deformation}

shear deformation!
Since stress and strain have components in several directions, we need a notation to represent these (as you learnt initially in Unified)

Several possible

- Tensor (indicial) notation
- Contracted notation
- Engineering notation
- Matrix notation

**IMPORTANT:** Regardless of the notation, the equations and concepts have the same meaning

⇒ learn, be comfortable with, be able to use all notations

**Tensor (or Summation) Notation**

- “Easy” to write complicated formulae
- “Easy” to mathematically manipulate
- “Elegant”, rigorous
- Use for derivations or to succinctly express a set of equations or a long equation
Example: \( x_i = f_{ij} y_j \)

- Rules for subscripts
  - Latin subscripts (m, n, p, q, …) take on the values 1, 2, 3 (3-D)
  - Greek subscripts (\( \alpha, \beta, \gamma \) …) take on the values 1, 2 (2-D)
  - When subscripts are repeated on one side of the equation within one term, they are called *dummy indices* and are to be summed on

  Thus:

  \[
  f_{ij} y_j = \sum_{j=1}^{3} f_{ij} y_j
  \]

  *But* \( f_{ij} y_j + g_i \) … *do not sum on i*!

- Subscripts which appear only once on the left side of the equation within one term are called *free indices* and represent a separate equation
Thus:

\[
x_i = \ldots
\]

\[\Rightarrow\]

\[
x_1 = \ldots
\]

\[
x_2 = \ldots
\]

\[
x_3 = \ldots
\]

**Key Concept:** The letters used for indices have no inherent meaning in and of themselves

Thus: \(x_i = f_{ij} y_j\)

is the same as: \(x_r = f_{rs} y_s\) or \(x_j = f_{ji} y_i\)

Now apply these concepts for stress/strain analysis:

1. **Coordinate System**
   Generally deal with right-handed rectangular Cartesian: \(y_m\)
Figure 3.4  Right-handed rectangular Cartesian coordinate system

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Engineering</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$y$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$z$</td>
</tr>
</tbody>
</table>

Note: *Normally* this is so, but always check definitions in any article, book, report, etc. Key issue is self-consistency, not consistency with a worldwide standard (an official one does not exist!)
2. **Deformations/Displacements** (3)

*Figure 3.5*

- \( p(y_1, y_2, y_3) \), small \( p \) (deformed position)
- \( P(Y_1, Y_2, Y_3) \) (original position)

\[
\mathbf{u}_m = p(y_m) - P(y_m)
\]

--> **Compare notations**

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Engineering</th>
<th>Direction in Engineering</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 )</td>
<td>( u )</td>
<td>( x )</td>
</tr>
<tr>
<td>( u_2 )</td>
<td>( v )</td>
<td>( y )</td>
</tr>
<tr>
<td>( u_3 )</td>
<td>( w )</td>
<td>( z )</td>
</tr>
</tbody>
</table>
3. **Components of Stress** (6)

\( \sigma_{mn} \) “Stress Tensor”

2 subscripts ⇒ 2nd order tensor

6 independent components

**Extensional**

\[
\begin{array}{c}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{array}
\]

**Shear**

\[
\begin{array}{c}
\sigma_{12} = \sigma_{21} \\
\sigma_{23} = \sigma_{32} \\
\sigma_{13} = \sigma_{31}
\end{array}
\]

**Note:** stress tensor is symmetric

\( \sigma_{mn} = \sigma_{nm} \)

due to *equilibrium* (moment) considerations

**Meaning of subscripts:**

- \( \sigma_{mn} \) stress acts in n-direction
- stress acts on face with normal vector in the m-direction
Figure 3.6  Differential element in rectangular system

\[ \text{NOTE: If face has a “negative normal”, positive stress is in negative direction} \]

--> Compare notations

<table>
<thead>
<tr>
<th>Tensor</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{11}$</td>
<td>$\sigma_x$</td>
</tr>
<tr>
<td>$\sigma_{22}$</td>
<td>$\sigma_y$</td>
</tr>
<tr>
<td>$\sigma_{33}$</td>
<td>$\sigma_z$</td>
</tr>
<tr>
<td>$\sigma_{23}$</td>
<td>$\sigma_{yz}$</td>
</tr>
<tr>
<td>$\sigma_{13}$</td>
<td>$\sigma_{xz}$</td>
</tr>
<tr>
<td>$\sigma_{12}$</td>
<td>$\sigma_{xy}$</td>
</tr>
</tbody>
</table>

\[ = \tau_{yz}, \quad = \tau_{xz}, \quad = \tau_{xy} \]

sometimes used for shear stresses
4. **Components of Strain**  

\[ \varepsilon_{mn} \quad \text{“Strain Tensor”} \]

2 subscripts \( \Rightarrow \) 2nd order tensor

6 independent components

**Extensional**

\[
\begin{array}{ccc}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33}
\end{array}
\]

**Shear**

\[
\begin{array}{ccc}
\varepsilon_{12} = \varepsilon_{21} \\
\varepsilon_{23} = \varepsilon_{32} \\
\varepsilon_{13} = \varepsilon_{31}
\end{array}
\]

**NOTE** (again): strain tensor is symmetric  

\[ \varepsilon_{mn} = \varepsilon_{nm} \]

due to geometrical considerations

(from Unified)
Meaning of subscripts *not* like stress

\[ \varepsilon_{mn} \]

\( m = n \Rightarrow \) extension along \( m \)
\( m \neq n \Rightarrow \) rotation in \( m-n \) plane

**BIG DIFFERENCE** for strain tensor:
There is a difference in the shear components of strain between tensor and engineering (unlike for stress).

**Figure 3.7**  
Representation of shearing of a 2-D element

[Image: Representation of shearing of a 2-D element]
total angular change = $\phi_{12} = \varepsilon_{12} + \varepsilon_{21} = 2\varepsilon_{12}$

(recall that $\varepsilon_{12}$ and $\varepsilon_{21}$ are the same due to geometrical considerations)

But, engineering shear strain is the total angle: $\phi_{12} = \varepsilon_{xy} = \gamma_{xy}$

-->

Comparison of notations

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>$\varepsilon_{11}$</td>
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<tr>
<td>$\varepsilon_{22}$</td>
<td>$\varepsilon_y$</td>
</tr>
<tr>
<td>$\varepsilon_{33}$</td>
<td>$\varepsilon_z$</td>
</tr>
<tr>
<td>$2\varepsilon_{23} = \varepsilon_{yz}$</td>
<td>$\gamma_{yz}$</td>
</tr>
<tr>
<td>$2\varepsilon_{13} = \varepsilon_{xz}$</td>
<td>$\gamma_{xz}$</td>
</tr>
<tr>
<td>$2\varepsilon_{12} = \varepsilon_{xy}$</td>
<td>$\gamma_{xy}$</td>
</tr>
</tbody>
</table>

Thus, factor of 2 will pop up

When we consider the equations of elasticity, the 2 comes out naturally.

(But, remember this “physical” explanation)
When dealing with shear strains, must know if they are tensorial or engineering…DO NOT ASSUME!

5. **Body Forces** (3)

   $f_i$ internal forces act along axes

   (resolve them in this manner -- can always do that)

   --> **Compare notations**

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>$f_x$</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$f_y$</td>
</tr>
<tr>
<td>$f_3$</td>
<td>$f_z$</td>
</tr>
</tbody>
</table>
6. **Elasticity Tensor** (? ... will go over later)

\[ E_{mnqp} \] relates stress and strain

(we will go over in detail, ... recall introduction in Unified)

### Other Notations

**Engineering Notation**

- One of two most commonly used
- Requires writing out all equations (no “shorthand”)
- Easier to see all components when written out fully

**Contracted Notation**

- Other of two most commonly used
- Requires less writing
- Often used with composites (“reduces” four subscripts on elasticity term to two)
- Meaning of subscripts not as “physical”
- Requires writing out all equations generally (there is contracted “shorthand”)

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--> subscript changes

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Engineering</th>
<th>Contracted</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>x</td>
<td>1</td>
</tr>
<tr>
<td>22</td>
<td>y</td>
<td>2</td>
</tr>
<tr>
<td>33</td>
<td>z</td>
<td>3</td>
</tr>
<tr>
<td>23, 32</td>
<td>yz</td>
<td>4</td>
</tr>
<tr>
<td>13, 31</td>
<td>xz</td>
<td>5</td>
</tr>
<tr>
<td>12, 21</td>
<td>xy</td>
<td>6</td>
</tr>
</tbody>
</table>

--> Meaning of “4, 5, 6” in contracted notation
- Shear component
- Represents axis (x_n) “about which” shear rotation takes place via:

\[ m = 3 + n \]

\[ \gamma_m \]

or

\[ \tau_m \]

\[ x_n \]

**Figure 3.8** Example:
Rotation about y_3

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Matrix notation

• “Super” shorthand
• Easy way to represent system of equations
• Especially adaptable with indicial notation
• Very useful in manipulating equations (derivations, etc.)

Example:

\[ x_i = A_{ij} y_j \]

\[ \tilde{x} = \tilde{A} \tilde{y} \]

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

(Will see a little of this … mainly in 16.21)

KEY: Must be able to use various notations. Don’t rely on notation, understand concept that is represented.