Principle of minimum potential energy

The principle of virtual displacements applies regardless of the constitutive law. Restrict attention to elastic materials (possibly nonlinear). Start from the PVD:

\[
\int_V \sigma_{ij} \epsilon_{ij} dV = \int_S t_i \bar{u}_i dS + \int_V f_i \bar{u}_i dV, \quad \forall \bar{u}/\bar{\bar{u}} = 0 \text{ on } S_u
\]  

(1)

Replacing the expression for the stresses for elastic materials:

\[
\sigma_{ij} = \frac{\partial U_0}{\partial \epsilon_{ij}}
\]

and assuming that the virtual displacement field is a variation of the equilibrated displacement field $\bar{u} = \delta u$, $\epsilon_{ij} = \delta \epsilon_{ij}$.

\[
\int_V \frac{\partial U_0}{\partial \epsilon_{ij}} \delta \epsilon_{ij} dV = \int_S t_i \delta u_i dS + \int_V f_i \delta u_i dV
\]
The expression over the brace is the variation of the strain energy density \( \delta U_0 \):
\[
\delta U_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}
\]

Using the properties of calculus of variations \( \delta \int() = \int \delta() \):
\[
\int \delta U_0 dV = \delta \int U_0 dV = \delta U = \delta \left( \int_S t_i u_i dS + \int_V f_i u_i dV \right) = \delta (-V)
\]
where \( V \) is the potential of the external loads. Therefore:

\[
\delta \Pi = \delta (U + V) = 0
\]

which is known as the \textit{Principle of minimum potential energy} (PMPE).

Let’s take the reverse path. Starting from the potential energy:

\[
\Pi(u_i) = \int_V \frac{1}{2} C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} dV - \int_S t_i u_i dS - \int_V f_i u_i dV
\]

we would like to apply our tools of calculus of variations to find the extrema of \( \Pi \):

\[
\delta_{u_i} \Pi = 0 = \int_V \frac{1}{2} C_{ijkl} (\delta \varepsilon_{kl} \varepsilon_{ij} + \varepsilon_{kl} \delta \varepsilon_{ij}) dV - \int_S t_i \delta u_i dS - \int_V f_i \delta u_i dV
\]

and, by symmetry of \( C_{ijkl} \):

\[
\int_V C_{ijkl} \varepsilon_{kl} \delta \varepsilon_{ij} dV = \int_S t_i \delta u_i dS + \int_V f_i \delta u_i dV
\]

Note that this is the expression of the Principle of Virtual Displacements applied to a linear elastic material.

In fact the expression of the PMPE we derived by setting the variations of \( \Pi = 0 \) only says that \( \Pi \) is stationary with respect to variations in the displacement field when the body is in equilibrium.

We can prove that it is indeed a minimum in the case of a linear elastic material: \( U_0 = \frac{1}{2} C_{ijkl} \varepsilon_{kl} \). We want to show:

\[
\Pi(v) \geq \Pi(u), \ \forall v
\]

\[
\Pi(v) = \Pi(u) \iff v = u
\]
Consider $\bar{u} = u + \delta u$:

$$\Pi(u + \delta u) = \int_V \left[ \frac{1}{2} C_{ijkl}(\epsilon_{ij} + \delta \epsilon_{ij})(\epsilon_{kl} + \delta \epsilon_{kl}) \right] dV$$

$$- \int_S t_i(u_i + \delta u_i) dS - \int_V F_i(u_i + \delta u_i) dV$$

$$= \Pi(u) + \mathcal{P} \int_V \frac{1}{2} C'_{ijkl} \delta \epsilon_{ij} \delta \epsilon_{kl} dV + \int_V \frac{1}{2} C_{ijkl} \delta \epsilon_{ij} \delta \epsilon_{kl} dV$$

$$- \int_S t_i \delta u_i dS - \int_V f_i \delta u_i dV$$

The second, fourth and fifth term disappear after invoking the PVD and we are left with:

$$\Pi(u + \delta u) = \Pi(u) + \int_V \frac{1}{2} C_{ijkl} \delta \epsilon_{ij} \delta \epsilon_{kl} dV$$

The integral is always $\geq 0$, since $C_{ijkl}$ is positive definite. Therefore:

$$\Pi(u + \delta u) = \Pi(u) + a, \quad a \geq 0, \quad a = 0 \iff \delta u = 0$$

and

$$\Pi(v) \geq \Pi(u), \quad \forall v$$

$$\Pi(v) = \Pi(u) \iff v = u$$

as sought.
Castigliano’s First theorem

Given a body in equilibrium under the action of $N$ concentrated forces $F_I$. The potential energy of the external forces is given by:

$$V = -\sum_{I=1}^{N} F_I u_I$$

where the $u_I$ are the values of the displacement field at the point of application of the forces $F_I$. Imagine that somehow we can express the strain energy as a function of the $u_I$, i.e.:

$$U = U(u_1, u_2, \ldots, u_N) = U(u_I)$$

Then:

$$\Pi = \Pi(u_I) = U(u_I) + V = U(u_I) - \sum_{I=1}^{N} F_I u_I$$
Invoking the PMPE:

$$\delta \Pi = 0 = \frac{\partial U}{\partial u_I} \delta u_I - \sum_{I=1}^{N} F_I \frac{\partial u_I}{\partial u_J} \delta u_J$$

$$= \frac{\partial U}{\partial u_I} \delta u_I - \sum_{I=1}^{N} F_I \delta u_I \delta u_J$$

$$= \frac{\partial U}{\partial u_I} \delta u_I - \sum_{I=1}^{N} F_I \delta u_I$$

$$= \left( \frac{\partial U}{\partial u_I} - F_I \right) \delta u_I$$

$$\forall \delta u_I \iff F_I = \frac{\partial U}{\partial u_I}$$

Theorem: If the strain energy can be expressed in terms of \( N \) displacements corresponding to \( N \) applied forces, the first derivative of the strain energy with respect to displacement \( u_I \) is the applied force.

Example:

$$\varepsilon_I = \sqrt{\frac{(L + u)^2 + v^2}{L^2}} - 1 \sim \frac{u}{L}$$

$$\varepsilon_I I = \sqrt{\frac{(L + u)^2 + (L - v)^2}{2L^2}} - 1 \sim \frac{1}{2} \frac{u - v}{L}$$
\[
U = \frac{1}{2}\left\{AE\left(\frac{u}{L}\right)^2 + AE\sqrt{2L}\left[\frac{1}{2}\left(\frac{u-v}{L}\right)\right]^2\right\}
\]

Note that we have written \(U = U(u, v)\). According to the theorem:

\[
0 = \frac{\partial U}{\partial u}
\]

\[
F = \frac{\partial U}{\partial v}
\]

See solution in accompanying mathematica file.