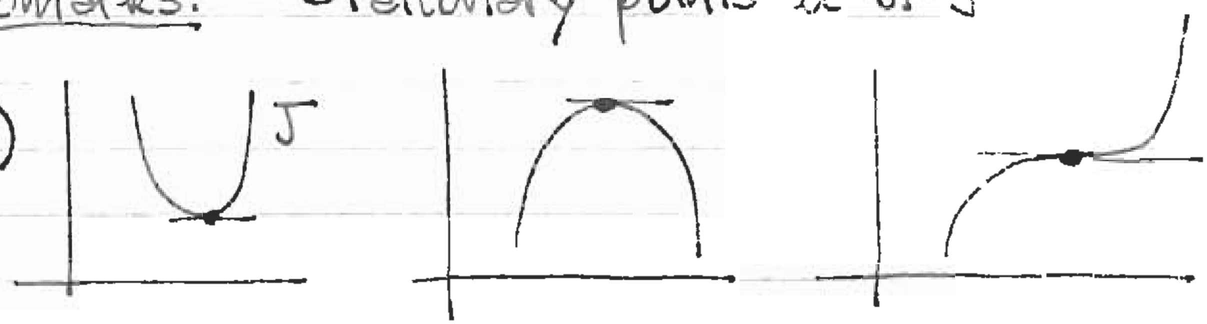
Remarks: Stationary points $u$ of $J$

i) 


ii) $\underbrace{\langle DJ(u), \eta \rangle}_{G(u,\eta)} = 0$

$G(u, \eta) = 0$

When is $G(u, \eta)$ the first variation of a functional $J(u)$?

Theorem (Vainberg): There is a functional

$J(u)$ s.t. $\langle DJ(u), \eta \rangle = G(u,\eta)$ iff

$$\langle DG(u,\eta), \xi \rangle = \langle DG(u,\xi), \eta \rangle$$

(reciprocity)

If this condition is satisfied, then

$$J(u) = \int_0^1 G(tu, u)\, dt$$

* Exercise: $G(u,\eta) = \int_B \left[ \frac{\partial F}{\partial u_i} \eta_i + \frac{\partial F}{\partial u_{i,j}} \eta_{i,j} \right] dV - \int_{S_2} \frac{\partial \phi}{\partial u_i} \eta_i \, ds$

Show reciprocity and obtain original functional

* Linear equations: $A_{ij} u_j + f_i = 0$

## Apply to L.E

$$\begin{aligned}
&\sigma_{ij,j} + f_i = 0 && \text{in } B \\
&\sigma_{ij} n_j = \bar{t}_i && \text{on } S_2 \\
&\varepsilon_{ij} = u_{(i,j)} && \text{in } B \\
&u_i = \bar{u}_i && \text{on } S_1 \\
&\sigma_{ij}(\varepsilon) = \frac{\partial W}{\partial \varepsilon_{ij}} && \text{in } B
\end{aligned}$$

$$u \rightarrow \begin{Bmatrix} u \\ \varepsilon \\ \sigma \end{Bmatrix} \qquad \eta \rightarrow \begin{Bmatrix} \eta \\ \alpha \\ \beta \end{Bmatrix}$$

Write field equations in integral form (weighted average sense)

Signs!

$$\int_B \left\{ -(\sigma_{ij,j} + f_i)\eta_i + [\sigma_{ij}(\varepsilon) - \sigma_{ij}]\alpha_{ij} + (u_{(i,j)} - \varepsilon_{ij})\beta_{ij} \right\} dV$$

$$- \int_{S_1} (u_i - \bar{u}_i)\beta_{ij} n_j \, ds + \int_{S_2} (\sigma_{ij} n_j - \bar{t}_i)\eta_i \, ds = 0$$

i.e. $G((u, \varepsilon, \sigma); (\eta, \alpha, \beta)) = 0$

Integrate by parts to obtain "weak form"

$$\int_B \sigma_{ij}\eta_{(i,j)} - f_i\eta_i + (\sigma_{ij}(\varepsilon) - \sigma_{ij})\alpha_{ij} + (u_{(i,j)} - \varepsilon_{ij})\beta_{ij} \, dV -$$

$$\underbrace{-\int_{S_1}(u_i-\bar{u}_i)\,\beta_{ij}n_j\,ds+\int_{S_2}(\sigma_{ij}n_j-\bar{t}_i)\,\eta_i\,ds-\int_S\sigma_{ij}n_j\eta_i\,ds}$$

$$-\int_{S_1}\left[(u_i-\bar{u}_i)\beta_{ij}+\sigma_{ij}\eta_i\right]n_j\,ds-\int_{S_2}\bar{t}_i\eta_i\,ds$$

$$\langle G((u,\varepsilon,\sigma),(\eta,\alpha,\beta))\rangle=\int_B\Big[\sigma_{ij}\,\eta_{(i,j)}-f_i\eta_i+(\sigma_{ij}(\varepsilon)-\sigma_{ij})\alpha_{ij}$$

$$+\,(u_{(i,j)}-\varepsilon_{ij})\beta_{ij}\Big]dv-\int_{S_1}\left[(u_i-\bar{u}_i)\beta_{ij}+\sigma_{ij}\eta_i\right]n_j\,ds-\int_{S_2}\bar{t}_i\eta_i\,ds$$

Does it derive from a ~~potential~~ functional?

$$\langle DG((u,\varepsilon,\sigma),(\eta,\alpha,\beta)),(\eta',\alpha',\beta')\rangle\stackrel{?}{=}$$

$$\langle DG((u,\varepsilon,\sigma),(\eta',\alpha',\beta')),(\eta,\alpha,\beta)\rangle$$

$\eta,\alpha,\beta$ not varied

$$\text{lhs: }\int_B\Big[\underbrace{\beta'_{ij}\eta_{(i,j)}}_{A}-\overset{\delta(f_i\eta_i)}{0}+\Big(\frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}}(\varepsilon)\,\alpha'_{kl}-\underbrace{\beta'_{ij}}_{B}\Big)\alpha_{ij}\,+$$

$$+\,\Big(\underbrace{\eta'_{(i,j)}}_{A'}-\underbrace{\alpha'_{ij}}_{B'}\Big)\beta_{ij}\Big]dv-\int_{S_1}\Big(\underbrace{\eta'_i\beta_{ij}}_{C}+\underbrace{\beta'_{ij}\eta_i}_{C'}\Big)n_j\,ds-0$$

reciprocity $\Longrightarrow$ $\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \alpha'_{kl} \alpha_{ij} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \alpha_{kl} \alpha'_{ij}$

$$= \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} \alpha'_{kl} \alpha_{ij} \Longleftrightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$$

$$\Longleftrightarrow \quad \underline{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}(\varepsilon)}$$

Obtain $J(u, \varepsilon, \sigma)$ using Vainberg's recipe:

$$J(u,\varepsilon,\sigma) = \int_0^1 G((tu, t\varepsilon, t\sigma), (u,\varepsilon,\sigma))\, dt$$

$$= \int_0^1 \Big\{ \int_B \big[ t\sigma_{ij} u_{(i,j)} - f_i u_i + (\sigma_{ij}(t\varepsilon) - t\sigma_{ij})\varepsilon_{ij} + (t u_{(i,j)} - t\varepsilon_{ij})\sigma_{ij} \big] dv$$

$$- \int_{S_1} [(tu_i - \bar{u}_i)\sigma_{ij} + t\sigma_{ij} u_i] n_j\, ds - \int_{S_2} \bar{t}_i u_i\, ds \Big\}\, dt$$

$$= \int_B \Big[ \cancel{\tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}} - f_i u_i + \int_0^1 \frac{\partial W}{\partial t}(t\varepsilon)\, dt - \cancel{\tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}} + \frac{1}{2} u_{(i,j)}\sigma_{ij}$$

$$- \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \Big] dv - \int_{S_1} \sigma_{ij} n_j (u_i - \bar{u}_i)\, ds - \int_{S_2} \bar{t}_i u_i\, ds$$

$$= \int_B W(\varepsilon) - f_i u_i + \frac{1}{2}\sigma_{ij}(u_{(i,j)} - \varepsilon_{ij})\, dv - \int_{S_1} \sigma_{ij} n_j (u_i - \bar{u}_i)\, ds - \int_{S_2} \bar{t}_i u_i\, ds$$

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$$J(u,\varepsilon,h) = \int_B \left[ W(\varepsilon) - f_i u_i + \sigma_{ij}\left(u_{(i,j)} - \varepsilon_{ij}\right) \right] dv - \int_{S_1} \sigma_{ij} n_j \left(u_i - \bar{u}_i\right) ds$$

$$- \int_{S_2} \bar{t}_i u_i \, ds$$

Hu-Washizu functional