(1) \[ B = B_h = \bigcup_{e=1}^{E} \Omega^e \]

(2) Use local interpolation of \( u_h \)

\[ u_h^e = \text{restriction of } u_h \text{ to } \Omega^e \]

\[ u_h^e = \sum_{a=1}^{n} u_a^e N_a^e(x) \]

\[ N_a^e(x^e) = \delta_{ab} \]

Global interpolation: Local element nodes must "fit" together and define global nodes.

- **Continuity requirements**

  Global "\( u_h \)" must be in \( H^1(B) \)

**Math aside:**

Need to measure "size" of functions (errors in particular) \( \rightarrow \) norms and seminorms.

Natural norms to use in problems such as linear elasticity:

- Sobolev norms
Need convenient way to express partial derivatives
multi-indices

Multi-index "\( \alpha \)" of dimension "\( d \)" is an array of nonnegative indices:

\[ \{ \alpha_1, \alpha_2, \ldots, \alpha_d \} \]

The degree \( |\alpha| \) of the multi-index is the sum

\[ |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d \]

**Definition:** \( u : \mathbb{R}^d \to \mathbb{R} \)

\[ D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}} \]

**Definition:** Semi-norm:

Let \( \Omega \subset \mathbb{R}^d \) an open bounded set, \( m \geq 0 \), \( 1 \leq p < \infty \)

\( u : \Omega \to \mathbb{R} \) \( m \)-times continuously differentiable in \( \Omega \).

\( (C^m(\Omega)) \)

\[ \| u \|_{m,p} = \left( \sum_{|\alpha| = m} \left( \int_{\Omega} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} \right) \]
Definition: \( \Omega \in \mathbb{R}^d \) bounded open set, \( m \geq 0, 1 \leq p < \infty \)
\( u : \Omega \to \mathbb{R} \) \( C^m(\Omega) \). Norm:
\[
\|u\|_{m,p} = \left( \sum_{k=0}^{m} |u|_{k,p}^p \right)^{1/p}
\]

Definition: \( W^{m,p}(\Omega) \) the Sobolev space of functions which can be obtained as limits of smooth functions under the norm \( \| \cdot \|_{m,p} \).

Roughly speaking, these limits may be thought as functions in \( L^p(\Omega) \) whose derivatives (in the distributional sense) up to order \( m \) are themselves in \( L^p(\Omega) \). In particular, the space \( W^{0,p} = L^p(\Omega) \) Lebesgue space.

Following standard practice, we shall denote \( H^m(\Omega) = W^{m,2}(\Omega) \).

The Sobolev space \( W^{m,p} \) is a complete normed space (Banach space).
In addition, $H^m(\Omega)$ are Hilbert spaces with the inner product:

$$\langle u, v \rangle_m = \sum_{1 \leq l \leq m} \int_\Omega D^l u \cdot D^l v \, dx$$

Went $u_h \in H^1_0(B)$ \(\Rightarrow\) $u = \overline{u}$ on $S$.

$H^1_0(B) = \{ u : B \subset \mathbb{R}^d \rightarrow \mathbb{R}^d / \| u \|_{1,2} < \infty, u = u_0$ on $S \}$

Think in terms of:

$$J(u) = \int_{B^2} \frac{1}{4} C_{ijkl} e_{ik} e_{ij} \, dv - \int_{B} f_i u_i \, dv - \int_{S} t_i u_i \, ds$$

$$\| u \|_E = \sqrt{a(u, u)} = \sqrt{\int_{B^2} C_{ijkl} e_{ik} e_{ij} \, dv}$$

**Conditions on $u_h^e$ (local interpolation)**

1. $N_h^e$ must be $C^1(\Omega_h^e)$ (sufficient, not necessary).

2. Global shape functions obtained by piecing together local shape functions must be $C^0$; derivatives may jump on a set of measure "0"
Shape functions $N_a^e$ must be uniquely defined on sides:

Global shape function:

$$u_h(x) = \sum_{e=1}^{E} u_h^e(x) = \sum_{e=1}^{E} \sum_{a=1}^{n} N_a^e(x) u_{ia}$$

(x not in boundary of elements)

Through connectivity map: $g(b,e) = a$

$$a = 1, \ldots, N$$
$$b = 1, \ldots, n$$
$$e = 1, \ldots, E$$

$$X_g(b,e) = X_a^e \quad \rightarrow \text{global/local mapping}$$

$$u_h(x) = \sum_{e=1}^{E} \sum_{b=1}^{n} N_b^e(x) u_g(b,e) = \sum_{a=1}^{N} u_a \cdot N_a(x)$$
\[ N_a = \sum_e N_a^e \]

**Support of \( N_a \): compact support**

\[ \text{support}(N_a) = \left\{ \sum_s CB / N_a(x) \neq 0 \forall x \in \Omega_s \right\} = \bigcup_{e} \Omega_e \text{ incident to node "a"} \]

**Computation of \( K \) and \( f \)**

\[ K_{\text{e,kl}} = \int_{B} C_{ijkl} N_{a,i}^e N_{b,j}^e dV \]

\[ = \int_{B} C_{ijkl} \left( \sum_{e=1}^{E} N_{a,i}^e \right) \left( \sum_{f=1}^{F} N_{b,j}^f \right) dV \]

\[ = \sum_{e=1}^{E} \int_{B} C_{ijkl} N_{a,i}^e N_{b,j}^e dV \]

\[ N_a^e \text{ compact support } \sum_{f} \sum_{e} = \sum_{e} ; \int_{B} = \int_{\Omega_{e}} \]

\[ = \sum_{e=1}^{E} \int_{\Omega_{e}} C_{ijkl} N_{a,i}^e N_{b,j}^e dV \]
\[ K_{ab} = \sum_{e=1}^{E} K_e \]

\[ \text{\# assembly operator} \]

Similarly:

\[ f_{ia} = \int_B f_i N_a \, dv + \text{treatment} \]

\[ = \int_B \{ \sum_{e} N_e \} \, dv = \sum_{e} \int_B N_e (f_{ia}^e) \, dv \]

\[ f_{ia} = \sum_{e} (f_{ia}^e) \]

\[ \text{\# assembly operation.} \]

Skip "\( B \)" matrix.

Isoparametric elements

Lagrangian family (quadrilaterals, hexahedra)


"The finite element method" T. J. R. Hughes, Dover 2000

"Linear static and dynamic analysis"

"The finite element method" O.C. Zienkiewicz, R. L. Taylor
5th edition, 2000
Define standard shape functions on standard domain (low order polynomials)

\[ \hat{N}_1(s_1, s_2) = \frac{1}{4} (1-s_1)(1-s_2) \]

\[ \hat{N}_2(s_1, s_2) = \frac{1}{4} (1+s_1)(1-s_2) \]

\[ \hat{N}_3(s_1, s_2) = \frac{1}{4} (1+s_1)(1+s_2) \]

\[ \hat{N}_4(s_1, s_2) = \frac{1}{4} (1-s_1)(1+s_2) \]

Verify:

i) \( \hat{N}_a(s_b) = S_{ab} \)

ii) Conformity (C0): restrictions of \( \hat{N}_a \) to element sides are linear

Define \( N_a(x_i) \): Isoparametric mapping