State-Space Systems

- What are state-space models?
- Why should we use them?
- How are they related to the transfer functions used in classical control design and how do we develop a state-space model?
- What are the basic properties of a state-space model, and how do we analyze these?
SS Introduction

- State space model: a representation of the dynamics of an $N^{th}$ order system as a first order differential equation in an $N$-vector, which is called the state.
- Convert the $N^{th}$ order differential equation that governs the dynamics into $N$ first-order differential equations.

- Classic example: second order mass-spring system

$$m\ddot{p} + c\dot{p} + kp = F$$

- Let $x_1 = p$, then $x_2 = \dot{p} = \dot{x}_1$, and

$$\dot{x}_2 = \ddot{p} = \frac{(F - c\dot{p} - kp)}{m} = \frac{(F - cx_2 - kx_1)}{m}$$

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

- Let $u = F$ and introduce the state

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \Rightarrow \dot{x} = Ax + Bu$$

- If the measured output of the system is the position, then we have that

$$y = p = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx$$
• Most general continuous-time linear dynamical system has form

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

where:

• \( t \in \mathbb{R} \) denotes time
• \( x(t) \in \mathbb{R}^n \) is the state (vector)
• \( u(t) \in \mathbb{R}^m \) is the input or control
• \( y(t) \in \mathbb{R}^p \) is the output

• \( A(t) \in \mathbb{R}^{n \times n} \) is the dynamics matrix
• \( B(t) \in \mathbb{R}^{n \times m} \) is the input matrix
• \( C(t) \in \mathbb{R}^{p \times n} \) is the output or sensor matrix
• \( D(t) \in \mathbb{R}^{p \times m} \) is the feedthrough matrix

• Note that the plant dynamics can be time-varying.
• Also note that this is a multi-input / multi-output (MIMO) system.

• We will typically deal with the time-invariant case
  ⇒ **Linear Time-Invariant (LTI)** state dynamics

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &=Cx(t) + Du(t)
\end{align*}
\]

so that now \( A, B, C, D \) are constant and do not depend on \( t \).
Basic Definitions

- **Linearity** – What is a linear dynamical system? A system \( G \) is linear with respect to its inputs and output

\[
u(t) \rightarrow G(s) \rightarrow y(t)
\]

iff superposition holds:

\[
G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 G u_1 + \alpha_2 G u_2
\]

So if \( y_1 \) is the response of \( G \) to \( u_1 \) \( (y_1 = G u_1) \), and \( y_2 \) is the response of \( G \) to \( u_2 \) \( (y_2 = G u_2) \), then the response to \( \alpha_1 u_1 + \alpha_2 u_2 \) is \( \alpha_1 y_1 + \alpha_2 y_2 \)

- A system is said to be **time-invariant** if the relationship between the input and output is independent of time. So if the response to \( u(t) \) is \( y(t) \), then the response to \( u(t - t_0) \) is \( y(t - t_0) \)

- Example: the system

\[
\begin{align*}
\dot{x}(t) &= 3x(t) + u(t) \\
y(t) &= x(t)
\end{align*}
\]

is LTI, but

\[
\begin{align*}
\dot{x}(t) &= 3t \ x(t) + u(t) \\
y(t) &= x(t)
\end{align*}
\]

is not.

- A matrix of second system is a function of absolute time, so response to \( u(t) \) will differ from response to \( u(t - 1) \).
• \( x(t) \) is called the **state of the system** at \( t \) because:
  • Future output depends only on current state and future input
  • Future output depends on past input only through current state
  • State summarizes effect of past inputs on future output – like the *memory of the system*

• **Example:** Rechargeable flashlight – the state is the *current state of charge* of the battery. If you know that state, then you do not need to know how that level of charge was achieved (assuming a perfect battery) to predict the future performance of the flashlight.
  • But to consider all nonlinear effects, you might also need to know how many cycles the battery has gone through
  • Key point is that you might expect a given linear model to accurately model the charge depletion behavior for a given number of cycles, but that model would typically change with the number cycles
Creating State-Space Models

- Most easily created from $N^{th}$ order differential equations that describe the dynamics
  - This was the case done before.
  - Only issue is which set of states to use – there are many choices.

- Can be developed from transfer function model as well.
  - Much more on this later

- Problem is that we have restricted ourselves here to linear state space models, and almost all systems are nonlinear in real-life.
  - Can develop linear models from nonlinear system dynamics
Equilibrium Points

- Often have a nonlinear set of dynamics given by

\[ \dot{x} = f(x, u) \]

where \( x \) is once again the state vector, \( u \) is the vector of inputs, and \( f(\cdot, \cdot) \) is a nonlinear vector function that describes the dynamics.

- First step is to define the point about which the linearization will be performed.

  - Typically about equilibrium points – a point for which if the system starts there it will remain there for all future time.

- Characterized by setting the state derivative to zero:

\[ \dot{x} = f(x, u) = 0 \]

  - Result is an algebraic set of equations that must be solved for both \( x_e \) and \( u_e \).

  - Note that \( \dot{x}_e = 0 \) and \( \dot{u}_e = 0 \) by definition.

  - Typically think of these nominal conditions \( x_e, u_e \) as “set points” or “operating points” for the nonlinear system.

- Example – pendulum dynamics: \( \ddot{\theta} + r \dot{\theta} + \frac{g}{l} \sin \theta = 0 \) can be written in state space form as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-rx_2 - \frac{g}{l} \sin x_1
\end{bmatrix}
\]

  - Setting \( f(x, u) = 0 \) yields \( x_2 = 0 \) and \( x_2 = -\frac{g}{rl} \sin x_1 \), which implies that \( x_1 = \theta = \{0, \pi\} \).
**Linearization**

- Typically assume that the system is operating about some nominal state solution $x_e$ (possibly requires a nominal input $u_e$)
  - Then write the actual state as $x(t) = x_e + \delta x(t)$ and the actual inputs as $u(t) = u_e + \delta u(t)$
  - The “$\delta$” is included to denote the fact that we expect the variations about the nominal to be “small”

- Can then develop the linearized equations by using the **Taylor series expansion** of $f(\cdot, \cdot)$ about $x_e$ and $u_e$.

- Recall the vector equation $\dot{x} = f(x, u)$, each equation of which
  \[ \dot{x}_i = f_i(x, u) \]
can be expanded as
  \[
  \frac{d}{dt}(x_{ei} + \delta x_i) = f_i(x_e + \delta x, u_e + \delta u) \\
  \approx f_i(x_e, u_e) + \left. \frac{\partial f_i}{\partial x} \right|_0 \delta x + \left. \frac{\partial f_i}{\partial u} \right|_0 \delta u
  \]
  where
  \[ \frac{\partial f_i}{\partial x} = \left[ \frac{\partial f_i}{\partial x_1} \ldots \frac{\partial f_i}{\partial x_n} \right] \]
  and $\cdot|_0$ means that we should evaluate the function at the nominal values of $x_e$ and $u_e$.

- The meaning of “small” deviations now clear – the variations in $\delta x$ and $\delta u$ must be small enough that we can ignore the higher order terms in the Taylor expansion of $f(x, u)$. 
• Since \( \frac{d}{dt} x_{e_i} = f_i(x_e, u_e) \), we thus have that
\[
\frac{d}{dt}(\delta x_i) \approx \left. \frac{\partial f_i}{\partial x} \right|_0 \delta x + \left. \frac{\partial f_i}{\partial u} \right|_0 \delta u
\]

• Combining for all \( n \) state equations, gives (note that we also set \( \approx \rightarrow \approx \) ) that
\[
\frac{d}{dt}\delta x = \begin{bmatrix}
\left. \frac{\partial f_1}{\partial x} \right|_0 \\
\left. \frac{\partial f_2}{\partial x} \right|_0 \\
\vdots \\
\left. \frac{\partial f_n}{\partial x} \right|_0
\end{bmatrix} \delta x + \begin{bmatrix}
\left. \frac{\partial f_1}{\partial u} \right|_0 \\
\left. \frac{\partial f_2}{\partial u} \right|_0 \\
\vdots \\
\left. \frac{\partial f_n}{\partial u} \right|_0
\end{bmatrix} \delta u
\]
\[
= A(t)\delta x + B(t)\delta u
\]

where
\[
A(t) \equiv \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_0
\]
\[
B(t) \equiv \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix}_0
\]
• Similarly, if the nonlinear measurement equation is \( y = g(x, u) \) and \( y(t) = y_e + \delta y \), then

\[
\delta y = \begin{bmatrix}
\frac{\partial g_1}{\partial x} \\
\vdots \\
\frac{\partial g_p}{\partial x}
\end{bmatrix}
\delta x + \begin{bmatrix}
\frac{\partial g_1}{\partial u} \\
\vdots \\
\frac{\partial g_p}{\partial u}
\end{bmatrix}
\delta u
\]

\[
= C(t)\delta x + D(t)\delta u
\]

• Typically drop the “\( \delta \)” as they are rather cumbersome, and (abusing notation) we write the state equations as:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

which is of the same form as the previous linear models.

• If the system is operating around just one set point then the partial fractions in the expressions for \( A-D \) are all constant \( \rightarrow \text{LTI linearized model.} \)
Stability of LTI Systems

• Consider a solution $x_s(t)$ to a differential equation for a given initial condition $x_s(t_0)$.
  
  • Solution is stable if other solutions $x_b(t_0)$ that start near $x_s(t_0)$ stay close to $x_s(t)$ $\forall t \Rightarrow$ stable in sense of Lyapunov (SSL).

  • If other solutions are SSL, but the $x_b(t)$ do not converge to $x_s(t)$ $\Rightarrow$ solution is neutrally stable.

  • If other solutions are SSL and $x_b(t) \to x(t)$ as $t \to \infty \Rightarrow$ solution is asymptotically stable.

  • A solution $x_s(t)$ is unstable if it is not stable.

• Note that a linear (autonomous) system $\dot{x} = Ax$ has an equilibrium point at $x_e = 0$
  
  • This equilibrium point is stable if and only if all of the eigenvalues of $A$ satisfy $\Re \lambda_i(A) \leq 0$ and every eigenvalue with $\Re \lambda_i(A) = 0$ has a Jordan block of order one.\(^1\)

  • Thus the stability test for a linear system is the familiar one of determining if $\Re \lambda_i(A) \leq 0$

• Somewhat surprisingly perhaps, we can also infer stability of the original nonlinear from the analysis of the linearized system model

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\(^1\)more on Jordan blocks on 6–77, but this basically means that these eigenvalues are not repeated.
• **Lyapunov’s indirect method**\(^2\) Let \(x_e = 0\) be an equilibrium point for the nonlinear autonomous system

\[
\dot{x}(t) = f(x(t))
\]

where \(f\) is continuously differentiable in a neighborhood of \(x_e\). Assume

\[
A = \left. \frac{\partial f}{\partial x} \right|_{x_e}
\]

Then:

- The origin is an asymptotically stable equilibrium point for the nonlinear system if \(\Re \lambda_i(A) < 0\ \forall \ i\)
- The origin is unstable if \(\Re \lambda_i(A) > 0\ \forall \ i\)

• Note that this doesn’t say anything about the stability of the nonlinear system if the linear system is neutrally stable.

• A very powerful result that is the basis of all linear control theory.

\(^2\)Much more on Lyapunov methods later too.
**Linearization Example**

- **Example:** simple spring. With a mass at the end of a linear spring (rate \( k \)) we have the dynamics

\[
    m\ddot{x} = -kx
\]

but with a “leaf spring” as is used on car suspensions, we have a nonlinear spring – the more it deflects, the stiffer it gets. Good model now is

\[
    m\ddot{x} = -k_1x - k_2x^3
\]

which is a “cubic spring”.

Fig. 1: Leaf spring from [http://en.wikipedia.org/wiki/Image:Leafs1.jpg](http://en.wikipedia.org/wiki/Image:Leafs1.jpg)

- Restoring force depends on deflection \( x \) in a nonlinear way.

Fig. 2: Response to linear \( k = 1 \) and nonlinear \((k_1 = k, k_2 = -2)\) springs (code at the end)
Consider the nonlinear spring with (set \( m = 1 \))

\[
\ddot{y} = -k_1 y - k_2 y^3
\]

gives us the nonlinear model \((x_1 = y \text{ and } x_2 = \dot{y})\)

\[
\frac{d}{dt} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} \Rightarrow \dot{x} = f(x)
\]

Find the equilibrium points and then make a state space model

For the equilibrium points, we must solve

\[
f(x) = \begin{bmatrix} \dot{y} \\ -k_1 y - k_2 y^3 \end{bmatrix} = 0
\]

which gives

\[
\dot{y}_e = 0 \quad \text{and} \quad k_1 y_e + k_2 (y_e)^3 = 0
\]

Second condition corresponds to \( y_e = 0 \) or \( y_e = \pm \sqrt{-k_1/k_2} \), which is only real if \( k_1 \) and \( k_2 \) are opposite signs.

For the state space model,

\[
A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}
\begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y_e)^2 & 0 \end{bmatrix}_0
\begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 (y_e)^2 & 0 \end{bmatrix}_0
\]

and the linearized model is \( \dot{x} = A \delta x \)
• For the equilibrium point $y_e = 0, \dot{y}_e = 0$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix}$$

which are the standard dynamics of a system with just a linear spring of stiffness $k_1$

• Stable motion about $y = 0$ if $k_1 > 0$

• Assume that $k_1 = -1, k_2 = 1/2$, then we should get an equilibrium point at $\dot{y} = 0, y = \pm \sqrt{2}$, and since $k_1 + k_2(y_e)^2 = 0$ then

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

which are the dynamics of a stable oscillator about the equilibrium point

Fig. 3: Nonlinear response ($k_1 = -1, k_2 = 0.5$). The figure on the right shows the oscillation about the equilibrium point.
function test=nlplant(ft);
global k1 k2
x0 = [-1 2]/10;
kl=1;k2=0;
[T,x]=ode23('plant', [0 20], x0); % linear
k2=-2;
[T1,x1]=ode23('plant', [0 20], x0); % nonlinear
figure(1);clf;
subplot(211)
plot(T,x(:,1),T1,x1(:,1),'--');
legend('Linear','Nonlinear')
ylabel('X','FontSize',ft)
xlabel('Time','FontSize',ft)
subplot(212)
plot(T,x(:,2),T1,x1(:,2),'--');
legend('Linear','Nonlinear')
ylabel('V','FontSize',ft)
xlabel('Time','FontSize',ft)
text(4,0.3,['k 2=',num2str(k2)],'FontSize',ft)
return
% use the following to call the function above
close all
set(0, 'DefaultAxesFontSize', 12, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 12, 'DefaultTextFontWeight','demi')
set(0,'DefaultAxesFontName','arial')
set(0,'DefaultAxesFontSize',12)
set(0,'DefaultTextFontName','arial')
set(gcf,'DefaultLineLineWidth',2);
set(gcf,'DefaultLineMarkerSize',10)
global k1 k2
nlplant(14)
print -f1 -dpng -r300 nlplant.png
k1=-1;k2=0.5;
% call plant.m
x0 = [sqrt(-k1/k2) .25];
[T,x]=ode23('plant', [0:.001:32], x0);
figure(1);subplot(211);plot(T,x(:,1));ylabel('y');xlabel('Time');grid
subplot(212);plot(T,x(:,2));ylabel('dy/dt');xlabel('Time');grid
figure(2);plot(x(:,1),x(:,2));grid
hold on;plot(x0(1),0,'rx','MarkerSize',20);hold off;
xlabel('y');ylabel('dy/dt')
axis([1.2 1.7 -.25 .25]);axis('square')
print -f1 -dpng -r300 nlplant2.png
print -f2 -dpng -r300 nlplant3.png

function [xdot] = plant(t,x);
% plant.m
global k1 k2
xdot(1) = x(2);
xdot(2) = -k1*x(1)-k2*(x(1))^3;
xdot = xdot';
Linearization Example: Aircraft Dynamics

- The basic dynamics are:
  \[ \vec{F} = m \ddot{\vec{c}} \quad \text{and} \quad \vec{T} = \dot{\vec{H}} \]
  \[ \Rightarrow \frac{1}{m} \vec{F} = \dot{\vec{c}}^B + B^I \vec{\omega} \times \vec{v}_c \quad \text{Transport Thm.} \]
  \[ \Rightarrow \vec{T} = \vec{H}^B + B^I \vec{\omega} \times \vec{H} \]

- Basic assumptions are:
  1. Earth is an inertial reference frame
  2. A/C is a rigid body
  3. Body frame \( B \) fixed to the aircraft \((\vec{i}, \vec{j}, \vec{k})\)

- Instantaneous mapping of \( \vec{v}_c \) and \( B^I \vec{\omega} \) into the body frame:
  \[ B^I \vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \quad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k} \]

  \[ \Rightarrow B^I \omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \]

- If \( x \) and \( z \) axes in plane of symmetry, can show that \( I_{xy} = I_{yz} = 0 \), but value of \( I_{xz} \) depends on specific body frame selected.
  - Instantaneous mapping of angular momentum
    \[ \vec{H} = H_x \vec{i} + H_y \vec{j} + H_z \vec{k} \]
    into the body frame given by
    \[
    H_B = \begin{bmatrix}
    H_x \\
    H_y \\
    H_z \\
    \end{bmatrix} = \begin{bmatrix}
    I_{xx} & 0 & I_{xz} \\
    0 & I_{yy} & 0 \\
    I_{xz} & 0 & I_{zz} \\
    \end{bmatrix} \begin{bmatrix}
    P \\
    Q \\
    R \\
    \end{bmatrix}
    \]
• The overall equations of motion are then:

$$\frac{1}{m} \ddot{\vec{F}} = \dot{\vec{q}}^B + B I \vec{\omega} \times \vec{v}_C$$

$$\Rightarrow \frac{1}{m} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\vec{T} = \dot{\vec{H}}^B + B I \vec{\omega} \times \vec{H}$$

$$\Rightarrow \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx} \dot{P} + I_{xz} \dot{R} \\ I_{yy} \dot{Q} \\ I_{zz} \dot{R} + I_{xz} \dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx} \dot{P} + I_{xz} \dot{R} + QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy} \dot{Q} + PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz} \dot{R} + I_{xz} \dot{P} + PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

• Equations are very nonlinear and complicated, and we have not even said where $\vec{F}$ and $\vec{T}$ come from $\Rightarrow$ need to linearize to develop analytic results

• Assume that the aircraft is flying in an *equilibrium condition* and we will linearize the equations about this nominal flight condition.
Linearization

- Can linearize about various steady state conditions of flight.
  - For steady state flight conditions must have
    \[ \vec{F} = \vec{F}_{\text{aero}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{thrust}} = 0 \quad \text{and} \quad T = 0 \]
    * So for equilibrium condition, forces balance on the aircraft
      \[ L = W \quad \text{and} \quad T = D \]
  - Also assume that \( \dot{P} = \dot{Q} = \dot{R} = \dot{U} = \dot{V} = \dot{W} = 0 \)
  - Impose additional constraints that depend on flight condition:
    * Steady wings-level flight \( \rightarrow \Phi = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = 0 \)

- Define the trim angular rates and velocities
  \[ B^I \omega^o_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \quad (v^o)_B = \begin{bmatrix} U^o \\ 0 \\ 0 \end{bmatrix} \]
  which are associated with the flight condition. In fact, these define the type of equilibrium motion that we linearize about. Note:
  - \( W_0 = 0 \) since we are using the stability axes, and
  - \( V_0 = 0 \) because we are assuming symmetric flight
- Proceed with linearization of the dynamics for various flight conditions

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<td>( \dot{\Psi} = \dot{\psi} )</td>
</tr>
</tbody>
</table>
• **Linearization for symmetric flight**

\[ U = U_0 + u, \quad V_0 = W_0 = 0, \quad P_0 = Q_0 = R_0 = 0. \]

Note that the forces and moments are also perturbed.

\[
\frac{1}{m} [X_0 + \Delta X] = \dot{U} + QW - RV \approx \dot{u} + qw - rv \approx \dot{u}
\]

\[
\frac{1}{m} [Y_0 + \Delta Y] = \dot{V} + RU - PW \\
\approx \dot{v} + r(U_0 + u) - pw \approx \dot{v} + rU_0
\]

\[
\frac{1}{m} [Z_0 + \Delta Z] = \dot{W} + PV - QU \approx \dot{w} + pv - q(U_0 + u) \\
\approx \dot{w} - qU_0
\]

\[
\Rightarrow \frac{1}{m} \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix} = \begin{bmatrix}
\dot{u} \\
\dot{v} + rU_0 \\
\dot{w} - qU_0
\end{bmatrix}
\]

\[
\Rightarrow \frac{1}{m} \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

• **Attitude motion:**

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} = \begin{bmatrix}
I_{xx} \dot{\hat{P}} + I_{xz} \dot{\hat{R}} + QR(I_{zz} - I_{yy}) + PQI_{xz} \\
I_{yy} \dot{\hat{Q}} + PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\
I_{zz} \dot{\hat{R}} + I_{xz} \dot{\hat{P}} + PQ(I_{yy} - I_{xx}) - QRI_{xz}
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
\Delta L \\
\Delta M \\
\Delta N
\end{bmatrix} = \begin{bmatrix}
I_{xx} \dot{\hat{p}} + I_{xz} \dot{\hat{r}} \\
I_{yy} \dot{\hat{q}} \\
I_{zz} \dot{\hat{r}} + I_{xz} \dot{\hat{p}}
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
\Delta L \\
\Delta M \\
\Delta N
\end{bmatrix} = \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
\]
• To understand equations in detail, and the resulting impact on the vehicle dynamics, we must investigate terms $\Delta X \ldots \Delta N$.

• We must also address the left-hand side $(\vec{F}, \vec{T})$

• **Net** forces and moments must be zero in equilibrium condition.
• Aerodynamic and Gravity forces are a function of equilibrium condition **AND** the perturbations about this equilibrium.

• Predict the changes to the aerodynamic forces and moments using a first order expansion in the key flight parameters

$$
\Delta X = \frac{\partial X}{\partial U} \Delta U + \frac{\partial X}{\partial W} \Delta W + \frac{\partial X}{\partial \dot{W}} \Delta \dot{W} + \frac{\partial X}{\partial \Theta} \Delta \Theta + \ldots + \frac{\partial X^g}{\partial \Theta} \Delta \Theta + \Delta X^c
$$

$$
= \frac{\partial X}{\partial U} u + \frac{\partial X}{\partial W} w + \frac{\partial X}{\partial \dot{W}} \dot{w} + \frac{\partial X}{\partial \Theta} \theta + \ldots + \frac{\partial X^g}{\partial \Theta} \theta + \Delta X^c
$$

• $\frac{\partial X}{\partial U}$ called **stability derivative** – evaluated at eq. condition.

• Clearly approximation since ignores lags in aerodynamics forces (assumes that forces only function of instantaneous values)
Stability Derivatives

• First proposed by Bryan (1911) – has proven to be a very effective way to analyze the aircraft flight mechanics – well supported by numerous flight test comparisons.

• The forces and torques acting on the aircraft are very complex nonlinear functions of the flight equilibrium condition and the perturbations from equilibrium.
  • Linearized expansion can involve many terms $u, \dot{u}, \ddot{u}, \ldots, w, \dot{w}, \ddot{w}, \ldots$
  • Typically only retain a few terms to capture the dominant effects.

• Dominant behavior most easily discussed in terms of the:
  • Symm. variables: $U, W, Q$ & forces/torques: $X, Z, M$
  • Asymm. variables: $V, P, R$ & forces/torques: $Y, L, N$

• Observation – for truly symmetric flight $Y, L, N$ will be exactly zero for any value of $U, W, Q$
  ⇒ Derivatives of asymmetric forces/torques with respect to the symmetric motion variables are zero.

• Further (convenient) assumptions:
  1. Derivatives of symmetric forces/torques with respect to the asymmetric motion variables are small and can be neglected.

  2. We can neglect derivatives with respect to the derivatives of the motion variables, but keep $\partial Z/\partial \dot{w}$ and $M_{\dot{w}} \equiv \partial M/\partial \dot{w}$ (aerodynamic lag involved in forming new pressure distribution on the wing in response to the perturbed angle of attack)

  3. $\partial X/\partial q$ is negligibly small.
\[ \frac{\partial X^g}{\partial \Theta} \bigg|_0 = -mg \cos \Theta_0 \quad \frac{\partial Z^g}{\partial \Theta} \bigg|_0 = -mg \sin \Theta_0 \]

- **Aerodynamic summary:**

1A \[ \Delta X = \left( \frac{\partial X}{\partial U} \right)_0 u + \left( \frac{\partial X}{\partial W} \right)_0 w \Rightarrow \Delta X \sim u, \alpha_x \approx w/U_0 \]

2A \[ \Delta Y \sim \beta \approx v/U_0, p, r \]

3A \[ \Delta Z \sim u, \alpha_x \approx w/U_0, \dot{\alpha}_x \approx \dot{w}/U_0, q \]

4A \[ \Delta L \sim \beta \approx v/U_0, p, r \]

5A \[ \Delta M \sim u, \alpha_x \approx w/U_0, \dot{\alpha}_x \approx \dot{w}/U_0, q \]

6A \[ \Delta N \sim \beta \approx v/U_0, p, r \]
• Result is that, with these force, torque approximations, equations 1, 3, 5 decouple from 2, 4, 6

• 1, 3, 5 are the **longitudinal dynamics** in $u, w, \text{and } q$

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} \dot{m} \\ m(\dot{w} - qU_0) \\ I_{yy}\dot{q} \end{bmatrix}$$

$$\approx \begin{bmatrix} (\frac{\partial X}{\partial U})_0 u + (\frac{\partial X}{\partial W})_0 w + (\frac{\partial X}{\partial \Theta})_0 \theta + \Delta X^c \\ (\frac{\partial Z}{\partial U})_0 u + (\frac{\partial Z}{\partial W})_0 w + (\frac{\partial Z}{\partial Q})_0 q + (\frac{\partial Z}{\partial \Theta})_0 \theta + \Delta Z^c \\ (\frac{\partial M}{\partial U})_0 u + (\frac{\partial M}{\partial W})_0 w + (\frac{\partial M}{\partial Q})_0 q + \Delta M^c \end{bmatrix}$$

• 2, 4, 6 are the **lateral dynamics** in $v, p, \text{and } r$

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} m(\dot{v} + rU_0) \\ I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix}$$

$$\approx \begin{bmatrix} (\frac{\partial Y}{\partial V})_0 v + (\frac{\partial Y}{\partial P})_0 p + (\frac{\partial Y}{\partial R})_0 r + \Delta Y^c \\ (\frac{\partial L}{\partial V})_0 v + (\frac{\partial L}{\partial P})_0 p + (\frac{\partial L}{\partial R})_0 r + \Delta L^c \\ (\frac{\partial N}{\partial V})_0 v + (\frac{\partial N}{\partial P})_0 p + (\frac{\partial N}{\partial R})_0 r + \Delta N^c \end{bmatrix}$$