Topic #6

16.30/31 Feedback Control Systems

State-Space Systems

• What are state-space models?
• Why should we use them?
• How are they related to the transfer functions used in classical control design and how do we develop a state-space model?
• What are the basic properties of a state-space model, and how do we analyze these?
TF’s to State-Space Models

- The goal is to develop a state-space model given a transfer function for a system $G(s)$.
  - There are many, many ways to do this.

- But there are three primary cases to consider:
  1. Simple numerator (strictly proper)
     \[ \frac{y}{u} = G(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} \]

  2. Numerator order less than denominator order (strictly proper)
     \[ \frac{y}{u} = G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{N(s)}{D(s)} \]

  3. Numerator equal to denominator order (proper)
     \[ \frac{y}{u} = G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \]

- These 3 cover all cases of interest
• Consider **case 1** (specific example of third order, but the extension to \(n^{th}\) follows easily)

\[
\frac{y}{u} = G(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}
\]

can be rewritten as the differential equation

\[
\ddot{y} + a_1 \dot{y} + a_2 y + a_3 y = u
\]

choose the output \(y\) and its derivatives as the state vector

\[
x = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}
\]

then the state equations are

\[
\dot{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} + [0]u
\]

• This is typically called the **controller form** for reasons that will become obvious later on.

• There are four classic (called **canonical**) forms – observer, controller, controllability, and observability. They are all useful in their own way.
• Consider case 2

\[
\frac{y}{u} = G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{N(s)}{D(s)}
\]

• Let

\[
\frac{y}{u} = \frac{y}{v} \cdot \frac{v}{u}
\]

where \(y/v = N(s)\) and \(v/u = 1/D(s)\)

• Then representation of \(v/u = 1/D(s)\) is the same as case 1

\[
\ddot{v} + a_1 \dot{v} + a_2 \dot{v} + a_3 v = u
\]

use the state vector

\[
\mathbf{x} = \begin{bmatrix} \dot{v} \\ \dot{v} \\ v \end{bmatrix}
\]

to get

\[
\dot{\mathbf{x}} = A_2 \mathbf{x} + B_2 u
\]

where

\[
A_2 = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\quad\text{and}\quad
B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

• Then consider \(y/v = N(s)\), which implies that

\[
y = b_1 \ddot{v} + b_2 \dot{v} + b_3 v
\]

\[
= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \ddot{v} \\ \dot{v} \\ v \end{bmatrix}
\]

\[
= C_2 \mathbf{x} + [0] u
\]
• Consider case 3 with

\[
\frac{y}{u} = G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}
\]

\[
= \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + a_1 s^2 + a_2 s + a_3} + D
\]

\[
= G_1(s) + D
\]

where

\[
D( s^3 + a_1 s^2 + a_2 s + a_3 )
\]

\[
+ ( +\beta_1 s^2 + \beta_2 s + \beta_3 )
\]

\[
= b_0 s^3 + b_1 s^2 + b_2 s + b_3
\]

so that, given the \( b_i \), we can easily find the \( \beta_i \)

\[
D = b_0
\]

\[
\beta_1 = b_1 - Da_1
\]

: 

• Given the \( \beta_i \), can find \( G_1(s) \)
  • Can make state-space model for \( G_1(s) \) as in case 2

• Then we just add the “feed-through” term \( Du \) to the output equation from the model for \( G_1(s) \)

• Will see that there is a lot of freedom in making a state-space model because we are free to pick the \( x \) as we want
Modal Form

- One particular useful canonical form is called the **Modal Form**
  - It is a diagonal representation of the state-space model.
- Assume for now that the transfer function has distinct real poles $p_i$ (easily extends to case with complex poles, see 7–??)

\[
G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s - p_1)(s - p_2)\cdots(s - p_n)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n}
\]
- Now define collection of first order systems, each with state $x_i$

\[
\begin{align*}
\frac{X_1}{U(s)} &= \frac{r_1}{s - p_1} \Rightarrow \dot{x}_1 = p_1 x_1 + r_1 u \\
\frac{X_2}{U(s)} &= \frac{r_2}{s - p_2} \Rightarrow \dot{x}_2 = p_2 x_2 + r_2 u \\
& \vdots \\
\frac{X_n}{U(s)} &= \frac{r_n}{s - p_n} \Rightarrow \dot{x}_n = p_n x_n + r_n u
\end{align*}
\]
- Which can be written as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

with

\[
A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix}, \quad B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T
\]
- Good representation to use for numerical robustness reasons.
  - Avoids some of the large coefficients in the other 4 canonical forms.
State-Space Models to TF’s

- Given the Linear Time-Invariant (LTI) state dynamics
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
  y(t) = Cx(t) + Du(t)
  \]
  what is the corresponding transfer function?

- Start by taking the Laplace Transform of these equations
  \[
  \mathcal{L}\{\dot{x}(t) = Ax(t) + Bu(t)\} \\
  sX(s) - x(0^-) = AX(s) + BU(s)
  \]
  \[
  \mathcal{L}\{y(t) = Cx(t) + Du(t)\} \\
  Y(s) = CX(s) + DU(s)
  \]
  which gives
  \[
  (sI - A)X(s) = BU(s) + x(0^-) \\
  \Rightarrow X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0^-)
  \]
  and
  \[
  Y(s) = [C(sI - A)^{-1}B + D] U(s) + C(sI - A)^{-1}x(0^-)
  \]

- By definition \( G(s) = C(sI - A)^{-1}B + D \) is called the **Transfer Function** of the system.

- And \( C(sI - A)^{-1}x(0^-) \) is the initial condition response.
  - It is part of the response, but not part of the transfer function.
SS to TF

- In going from the state space model
  \[ \dot{x}(t) = Ax(t) + Bu(t) \]
  \[ y(t) = Cx(t) + Du(t) \]

to the transfer function \( G(s) = C(sI - A)^{-1}B + D \) need to form inverse of matrix \((sI - A)\)

  - A symbolic inverse – not very easy.

- For simple cases, we can use the following:
  \[
  \begin{bmatrix}
  a_1 & a_2 \\
  a_3 & a_4 
  \end{bmatrix}^{-1} = \frac{1}{a_1a_4 - a_2a_3} \begin{bmatrix}
  a_4 & -a_2 \\
  -a_3 & a_1
  \end{bmatrix}
  \]

  For larger problems, we can also use Cramer’s Rule

- Turns out that an equivalent method is to form:\footnote{see here}
  \[
  G(s) = C(sI - A)^{-1}B + D = \frac{\det \begin{bmatrix}
  sI - A & -B \\
  C & D
  \end{bmatrix}}{\det(sI - A)}
  \]

  - Reason for this will become more apparent later (see 8–??) when we talk about how to compute the “zeros” of a state-space model (which are the roots of the numerator)

- **Key point:** System characteristic equation given by
  \[ \phi(s) = \det(sI - A) = 0 \]

  - It is the roots of \( \phi(s) = 0 \) that determine the poles of the system. Will show that these determine the time response of the system.
- Example from Case 2, page 6–4

\[
A = \begin{bmatrix}
-a_1 & -a_2 & -a_3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T
\]

then

\[
G(s) = \frac{1}{\det(sI - A)} \cdot \det(s + a_1 \quad a_2 \quad a_3 \\ -1 \quad s \quad 0 \quad 0 \\ 0 \quad -1 \quad s \quad 0 \\ b_1 \quad b_2 \quad b_3 \quad 0)
\]

\[
= \frac{b_3 + b_2 s + b_1 s^2}{\det(sI - A)}
\]

and \(\det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3\)

- Which is obviously the same as before.
State-Space Transformations

- State space representations are not unique because we have a lot of freedom in choosing the state vector.
  - Selection of the state is quite arbitrary, and not that important.
- In fact, given one model, we can transform it to another model that is equivalent in terms of its input-output properties.
- To see this, define Model 1 of $G(s)$ as
  \[
  \begin{align*}
  \dot{x}(t) &= Ax(t) + Bu(t) \\
  y(t) &= Cx(t) + Du(t)
  \end{align*}
  \]

- Now introduce the new state vector $z$ related to the first state $x$ through the transformation $x = Tz$
  - $T$ is an invertible (similarity) transform matrix
    \[
    \begin{align*}
    \dot{z} &= T^{-1}\dot{x} = T^{-1}(Ax + Bu) \\
    &= T^{-1}(ATz + Bu) \\
    &= (T^{-1}AT)z + T^{-1}Bu = \tilde{A}z + \tilde{B}u
    \end{align*}
    \]
    and
    \[
    y = Cx + Du = CTz + Du = \tilde{C}z + \tilde{D}u
    \]
- So the new model is
  \[
  \begin{align*}
  \dot{z} &= \tilde{A}z + \tilde{B}u \\
  y &= \tilde{C}z + \tilde{D}u
  \end{align*}
  \]
- Are these going to give the same transfer function? They must if these really are equivalent models.
Consider the two transfer functions:

\[ G_1(s) = C(sI - A)^{-1}B + D \]
\[ G_2(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \]

Does \( G_1(s) \equiv G_2(s) \)?

\[ G_1(s) = C(sI - A)^{-1}B + D \]
\[ = C(TT^{-1})(sI - A)^{-1}(TT^{-1})B + D \]
\[ = (CT) [T^{-1}(sI - A)^{-1}T] (T^{-1}B) + \bar{D} \]
\[ = (\bar{C}) [T^{-1}(sI - A)T]^{-1} (\bar{B}) + \bar{D} \]
\[ = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = G_2(s) \]

So the transfer function is not changed by putting the state-space model through a similarity transformation.

Note that in the transfer function

\[ G(s) = \frac{b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3} \]

we have 6 parameters to choose

But in the related state-space model, we have \( A - 3 \times 3, B - 3 \times 1, C - 1 \times 3 \) for a total of 15 parameters.

Is there a contradiction here because we have more degrees of freedom in the state-space model?

No. In choosing a representation of the model, we are effectively choosing a \( T \), which is also \( 3 \times 3 \), and thus has the remaining 9 degrees of freedom in the state-space model.