State-Space Systems

- **State-space model features**
- Observability
- Controllability
- Minimal Realizations


**State-Space Model Features**

- There are some key characteristics of a state-space model that we need to identify.
  - Will see that these are very closely associated with the concepts of pole/zero cancelation in transfer functions.

- **Example:** Consider a simple system
  \[ G(s) = \frac{6}{s + 2} \]
  for which we develop the state-space model
  
  Model # 1
  \[
  \begin{align*}
  \dot{x} &= -2x + 2u \\
  y &= 3x
  \end{align*}
  \]

- But now consider the new state space model \( \bar{x} = [x \ x_2]^T \)
  
  Model # 2
  \[
  \begin{align*}
  \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\
  y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{x}
  \end{align*}
  \]
  which is clearly different than the first model, and larger.

- But let’s looks at the transfer function of the new model:
  \[
  \bar{G}(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 3 & 0 \end{bmatrix} \left( \frac{1}{s + 2} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{6}{s + 2} \]

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This is a bit strange, because previously our figure of merit when comparing one state-space model to another (page 6–??) was whether they reproduced the same same transfer function

Now we have two very different models that result in the same transfer function

Note that I showed the second model as having 1 extra state, but I could easily have done it with 99 extra states!!

So what is going on?

A clue is that the dynamics associated with the second state of the model $x_2$ were eliminated when we formed the product

$$
\tilde{G}(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{s+2} \\ \frac{1}{s+1} \end{bmatrix}
$$

because the $A$ is decoupled and there is a zero in the $C$ matrix

Which is exactly the same as saying that there is a **pole-zero cancelation** in the transfer function $\tilde{G}(s)$

$$
\frac{6}{s+2} = \frac{6(s+1)}{(s+2)(s+1)} \triangleq \tilde{G}(s)
$$

Note that model #2 is one possible state-space model of $\tilde{G}(s)$ (has 2 poles)

For this system we say that the dynamics associated with the second state are **unobservable** using this sensor (defines $C$ matrix).

There could be a lot “motion” associated with $x_2$, but we would be unaware of it using this sensor.
There is an analogous problem on the input side as well. Consider:

Model # 1 \[
\begin{align*}
\dot{x} &= -2x + 2u \\
y &= 3x
\end{align*}
\]

with \( \bar{x} = [x \ x_2]^T \)

Model # 3 \[
\begin{align*}
\dot{\bar{x}} &= \begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix} \bar{x} + \begin{bmatrix}
2 \\
0
\end{bmatrix} u \\
y &= \begin{bmatrix}
3 & 2
\end{bmatrix} \bar{x}
\end{align*}
\]

which is also clearly different than model #1, and has a different form from the second model.

\[
\hat{G}(s) = \begin{bmatrix}
3 & 2
\end{bmatrix} \left(sI - \begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix}\right)^{-1} \begin{bmatrix}
2 \\
0
\end{bmatrix} \\
= \begin{bmatrix}
\frac{3}{s+2} & \frac{2}{s+1}
\end{bmatrix} \begin{bmatrix}
2 \\
0
\end{bmatrix} = \frac{6}{s + 2}
\]

Once again the dynamics associated with the pole at \( s = -1 \) are canceled out of the transfer function.

- But in this case it occurred because there is a 0 in the \( B \) matrix

So in this case we can “see” the state \( x_2 \) in the output \( C = \begin{bmatrix} 3 & 2 \end{bmatrix} \),

but we cannot “influence” that state with the input since

\[
B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]

So we say that the dynamics associated with the second state are uncontrollable using this actuator (defines the \( B \) matrix).
• Of course it can get even worse because we could have

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 3 & 0 \end{bmatrix} x
\end{align*}
\]

• So now we have

\[
\tilde{G}(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \left( sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{s+2} & 0 \\ 0 & \frac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2}
\]

• Get same result for the transfer function, but now the dynamics associated with \(x_2\) are both unobservable and uncontrollable.

• **Summary:** Dynamics in the state-space model that are **uncontrollable, unobservable**, or **both** do not show up in the transfer function.

• Would like to develop models that **only have** dynamics that are both **controllable** and **observable**

\[ \Rightarrow \text{ called a } \textbf{minimal realization} \]

• A state space model that has the lowest possible order for the given transfer function.

• But first need to develop tests to determine if the models are observable and/or controllable
Observability

- **Definition:** An LTI system is **observable** if the initial state $x(0)$ can be **uniquely deduced** from the knowledge of the input $u(t)$ and output $y(t)$ for all $t$ between $0$ and any finite $T > 0$.

  - If $x(0)$ can be deduced, then we can reconstruct $x(t)$ exactly because we know $u(t) \Rightarrow$ we can find $x(t) \forall t$.

  - Thus we need only consider the zero-input (homogeneous) solution to study observability.

$$y(t) = Ce^{At}x(0)$$

- This definition of observability is consistent with the notion we used before of being able to “see” all the states in the output of the decoupled examples

  - **ROT:** For those decoupled examples, if part of the state cannot be “seen” in $y(t)$, then it would be impossible to deduce that part of $x(0)$ from the outputs $y(t)$. 

• **Definition:** A state $x^* \neq 0$ is said to be **unobservable** if the zero-input solution $y(t)$, with $x(0) = x^*$, is zero for all $t \geq 0$

  • Equivalent to saying that $x^*$ is an unobservable state if
    \[ Ce^{At}x^* = 0 \quad \forall \quad t \geq 0 \]

• For the problem we were just looking at, consider Model #2 with $x^* = [0 \ 1]^T \neq 0$, then
  \[
  \dot{x} = \begin{bmatrix}
  -2 & 0 \\
  0 & -1
  \end{bmatrix} x + \begin{bmatrix}
  2 \\
  1
  \end{bmatrix} u \\
  y = \begin{bmatrix}
  3 & 0
  \end{bmatrix} \bar{x}
  \]

  so
  \[
  Ce^{At}x^* = \begin{bmatrix}
  3 & 0
  \end{bmatrix} \begin{bmatrix}
  e^{-2t} & 0 \\
  0 & e^{-t}
  \end{bmatrix} \begin{bmatrix}
  0 \\
  1
  \end{bmatrix}
  = \begin{bmatrix}
  3e^{-2t} & 0
  \end{bmatrix} \begin{bmatrix}
  0 \\
  1
  \end{bmatrix}
  = 0 \quad \forall \quad t
  \]

  **So,** $x^* = [0 \ 1]^T$ is an unobservable state for this system.

• But that is as expected, because we knew there was a problem with the state $x_2$ from the previous analysis
• **Theorem:** An LTI system is observable iff it has no unobservable states.
  - We normally just say that the pair \((A, C)\) is observable.

• **Pseudo-Proof:** Let \(x^* \neq 0\) be an unobservable state and compute the outputs from the initial conditions \(x_1(0)\) and \(x_2(0) = x_1(0) + x^*\)
  - Then
    \[
    y_1(t) = Ce^{At}x_1(0) \quad \text{and} \quad y_2(t) = Ce^{At}x_2(0)
    \]
  but
    \[
    y_2(t) = Ce^{At}(x_1(0) + x^*) = Ce^{At}x_1(0) + Ce^{At}x^* = Ce^{At}x_1(0) = y_1(t)
    \]
  - Thus 2 different initial conditions give the same output \(y(t)\), so it would be impossible for us to deduce the actual initial condition of the system \(x_1(t)\) or \(x_2(t)\) given \(y_1(t)\)

• Testing system observability by searching for a vector \(x(0)\) such that \(Ce^{At}x(0) = 0 \forall t\) is feasible, but very hard in general.
  - Better tests are available.
- **Theorem:** The vector $x^*$ is an unobservable state iff

$$\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix} x^* = 0$$

- **Pseudo-Proof:** If $x^*$ is an unobservable state, then by definition,

$$Ce^{At}x^* = 0 \quad \forall \ t \geq 0$$

But all the derivatives of $Ce^{At}$ exist and for this condition to hold, all derivatives must be zero at $t = 0$. Then

$$Ce^{At}x^* \bigg|_{t=0} = 0 \Rightarrow Cx^* = 0$$

$$\frac{d}{dt} Ce^{At}x^* \bigg|_{t=0} = 0 \Rightarrow Ca^{At}x^* \bigg|_{t=0} = CAx^* = 0$$

$$\frac{d^2}{dt^2} Ce^{At}x^* \bigg|_{t=0} = 0 \Rightarrow CA^2e^{At}x^* \bigg|_{t=0} = CA^2x^* = 0$$

$$\vdots$$

$$\frac{d^k}{dt^k} Ce^{At}x^* \bigg|_{t=0} = 0 \Rightarrow CA^ke^{At}x^* \bigg|_{t=0} = CA^kx^* = 0$$

- We only need retain up to the $n - 1^{th}$ derivative because of the *Cayley-Hamilton* theorem.
• **Simple test:** Necessary and sufficient condition for observability is that

\[
\text{rank } \mathcal{M}_o \triangleq \text{rank } \begin{bmatrix}
  C \\
  CA \\
  CA^2 \\
  \vdots \\
  CA^{n-1}
\end{bmatrix} = n
\]

• Why does this make sense?
  • The requirement for an unobservable state is that for \( x^* \neq 0 \)
    \[
    \mathcal{M}_o x^* = 0
    \]
  • Which is equivalent to saying that \( x^* \) is orthogonal to each row of \( \mathcal{M}_o \).
  • But if the rows of \( \mathcal{M}_o \) are considered to be vectors and these span the full \( n \)-dimensional space, then it is not possible to find an \( n \)-vector \( x^* \) that is orthogonal to each of these.
  • To determine if the \( n \) rows of \( \mathcal{M}_o \) span the full \( n \)-dimensional space, we need to test their linear independence, which is equivalent to the rank test\(^1\)

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\(^1\)Let \( M \) be a \( m \times p \) matrix, then the rank of \( M \) satisfies:
1. \( \text{rank } M \equiv \text{number of linearly independent columns of } M \)
2. \( \text{rank } M \equiv \text{number of linearly independent rows of } M \)
3. \( \text{rank } M \leq \min\{m, p\} \)