State-Space Systems

- **State-space model features**
- Controllability
Controllability

• **Definition:** An LTI system is **controllable** if, for every \(x^*(t)\) and every finite \(T > 0\), there exists an input function \(u(t)\), \(0 < t \leq T\), such that the system state goes from \(x(0) = 0\) to \(x(T) = x^*\).

• Starting at 0 is not a special case – if we can get to any state in finite time from the origin, then we can get from any initial condition to that state in finite time as well.\(^1\)

• This definition of controllability is consistent with the notion we used before of being able to “influence” all the states in the system in the decoupled examples (page 9–??).

• ROT: For those decoupled examples, if part of the state cannot be “influenced” by \(u(t)\), then it would be impossible to move that part of the state from 0 to \(x^*\).

• Need only consider the forced solution to study controllability.

\[
x_f(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau
\]

• Change of variables \(\tau_2 = t - \tau\), \(d\tau = -d\tau_2\) gives a form that is a little easier to work with:

\[
x_f(t) = \int_0^t e^{A\tau_2} Bu(t - \tau_2) d\tau_2
\]

• Assume system has \(m\) inputs.

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\(^1\)This controllability from the origin is often called reachability.
• Note that, regardless of the eigenstructure of $A$, the Cayley-Hamilton theorem gives

$$e^{At} = \sum_{i=0}^{n-1} A^i \alpha_i(t)$$

for some computable scalars $\alpha_i(t)$, so that

$$x_f(t) = \sum_{i=0}^{n-1} (A^iB) \int_0^t \alpha_i(\tau_2) u(t - \tau_2) d\tau_2 = \sum_{i=0}^{n-1} (A^iB) \beta_i(t)$$

for coefficients $\beta_i(t)$ that depend on the input $u(\tau), 0 < \tau \leq t$.

• Result can be interpreted as meaning that the state $x_f(t)$ is a linear combination of the $nm$ vectors $A^iB$ (with $m$ inputs).

• All linear combinations of these $nm$ vectors is the range space of the matrix formed from the $A^iB$ column vectors:

$$M_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

• **Definition:** Range space of $M_c$ is **controllable subspace** of the system

  • If a state $x_c(t)$ is not in the range space of $M_c$, it is not a linear combination of these columns $\Rightarrow$ it is impossible for $x_f(t)$ to ever equal $x_c(t)$ – called **uncontrollable state**.

• **Theorem:** LTI system is controllable iff it has no uncontrollable states.

  • Necessary and sufficient condition for controllability is that

  $$\text{rank } M_c \triangleq \text{rank } \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$$
Further Examples

• With Model \# 2:

\[
\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 3 & 0 \end{bmatrix} x
\]

\[
\mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix}
\]

\[
\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}
\]

• rank \( \mathcal{M}_0 = 1 \) and rank \( \mathcal{M}_c = 2 \)

• So this model of the system is controllable, but not observable.

• With Model \# 3:

\[
\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 3 & 2 \end{bmatrix} x
\]

\[
\mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -6 & -2 \end{bmatrix}
\]

\[
\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}
\]

• rank \( \mathcal{M}_0 = 2 \) and rank \( \mathcal{M}_c = 1 \)

• So this model of the system is observable, but not controllable.

• Note that controllability/observability are not intrinsic properties of a system. Whether the model has them or not depends on the representation that you choose.

• But they indicate that something else more fundamental is wrong...
Modal Tests

- Earlier examples showed the relative simplicity of testing observability/controllability for a system with a decoupled $A$ matrix.

- There is, of course, a very special decoupled form for the state-space model: the Modal Form (6–??)

- Assuming that we are given the model
  \[
  \begin{align*}
  \dot{x} &= Ax + Bu \\
  y &=Cx + Du
  \end{align*}
  \]
  and the $A$ is diagonalizable ($A = T\Lambda T^{-1}$) using the transformation
  \[
  T = \begin{bmatrix}
  | & & | \\
  v_1 & \cdots & v_n \\
  | & & | 
  \end{bmatrix}
  \]
  based on the eigenvalues of $A$. Note that we wrote:
  \[
  T^{-1} = \begin{bmatrix}
  - & w_1^T & - \\
  & \vdots & \\
  - & w_n^T & - 
  \end{bmatrix}
  \]
  which is a column of rows.

- Then define a new state so that $x = Tz$, then
  \[
  \begin{align*}
  \dot{z} &= T^{-1}\dot{x} = T^{-1}(Ax + Bu) \\
  &= (T^{-1}AT)z + T^{-1}Bu \\
  &= \Lambda z + T^{-1}Bu \\
  y &= Cx + Du \\
  &= CTz + Du
  \end{align*}
  \]
• The new model in the state $z$ is diagonal. There is no coupling in the dynamics matrix $\Lambda$.

• But by definition,

$$T^{-1}B = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} B$$

and

$$CT = C \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

• Thus if it turned out that

$$w_i^T B \equiv 0$$

then that element of the state vector $z_i$ would be \textit{uncontrollable} by the input $u$.

• Also, if

$$Cv_j \equiv 0$$

then that element of the state vector $z_j$ would be \textit{unobservable} with this sensor.

• Thus, all \textit{modes of the system are controllable and observable} if it can be shown that

$$w_i^T B \neq 0 \ \forall \ i$$

and

$$Cv_j \neq 0 \ \forall \ j$$
Cancelation

- Examples show the close connection between pole-zero cancelation and loss of observability and controllability. Can be strengthened.

- **Theorem:** The mode \((\lambda_i, v_i)\) of a system \((A, B, C, D)\) is unobservable iff the system has a zero at \(\lambda_i\) with direction \(\begin{bmatrix} v_i \\ 0 \end{bmatrix}\).

- **Proof:** If the system is unobservable at \(\lambda_i\), then we know

\[
(\lambda_iI - A)v_i = 0 \quad \text{It is a mode}
\]

\[
Cv_i = 0 \quad \text{That mode is unobservable}
\]

Combine to get:

\[
\begin{bmatrix} (\lambda_iI - A) \\ C \end{bmatrix} v_i = 0
\]

Or

\[
\begin{bmatrix} (\lambda_iI - A) & -B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ 0 \end{bmatrix} = 0
\]

which implies that the system has a zero at that frequency as well, with direction \(\begin{bmatrix} v_i \\ 0 \end{bmatrix}\).

- Can repeat the process looking for loss of controllability, but now using zeros with left direction \(\begin{bmatrix} w_i^T \\ 0 \end{bmatrix}\).
- **Combined Definition:** when a MIMO zero causes loss of either observability or controllability we say that there is a pole/zero cancellation.

- MIMO pole-zero (right direction generalized eigenvector) cancellation $\iff$ mode is unobservable

- MIMO pole-zero (left direction generalized eigenvector) cancellation $\iff$ mode is uncontrollable

- **Note:** This cancellation requires an agreement of both the frequency and the directionality of the system mode (eigenvector) and zero 
\[
\begin{bmatrix}
v_i \\
0
\end{bmatrix}
\text{ or } 
\begin{bmatrix}
w_i^T \\
0
\end{bmatrix}.
\]
Connection to Residue

- Recall that in modal form, the state-space model (assumes diagonalizable) is given by the matrices

\[
A = \begin{bmatrix}
p_1 & & \\
& \ddots & \\
p_n & & \end{bmatrix} \quad B = \begin{bmatrix} w_1^T B \\ \vdots \\ w_n^T B \end{bmatrix} \quad C = \begin{bmatrix} C_{v_1} & \cdots & C_{v_n} \end{bmatrix}
\]

for which case it can easily be shown that

\[
G(s) = C(sI - A)^{-1}B = \begin{bmatrix} C_{v_1} & \cdots & C_{v_n} \end{bmatrix} \begin{bmatrix} \frac{1}{s-p_1} \\ \vdots \\ \frac{1}{s-p_n} \end{bmatrix} \begin{bmatrix} w_1^T B \\ \vdots \\ w_n^T B \end{bmatrix} = \sum_{i=1}^{n} \frac{(C_{v_i})(w_i^T B)}{s - p_i}
\]

- Thus the residue of each pole is a direct function of the product of the degree of controllability and observability for that mode.
  - Loss of observability or controllability \( \Rightarrow \) residue is zero \( \Rightarrow \) that pole does not show up in the transfer function.
  - If modes have equal observability \( C_{v_i} \approx C_{v_j} \), but one \( w_i^T B \gg w_j^T B \)
    then the residue of the \( i \)th mode will be much larger.
  - Great way to approach model reduction if needed.
**Weaker Conditions**

- Often it is too much to assume that we will have full observability and controllability. Often have to make do with the following. System called:
  - **Detectable** if all unstable modes are *observable*
  - **Stabilizable** if all unstable modes are *controllable*

- So if you had a stabilizable and detectable system, there could be dynamics that you are not aware of and cannot influence, but you know that they are at least stable.

- That is enough information on the system model for now – will assume minimal models from here on and start looking at the control issues.