Recitation 12
16.30/31: Estimation and Control of Aerospace Systems
November 29, 2010

Consider a space telescope, and let \( \vec{d} \) be a unit vector aligned with the telescope’s line of sight. It is desired to point the telescope towards a star, in direction \( \vec{d}_0 \). The dynamics of the spacecraft are described by the equations of motion

\[
J\ddot{\omega} + \omega \times J\omega = \tau,
\]

where \( \omega \) is the angular velocity of the spacecraft (in body axes), \( J \) is its inertia tensor, and \( \tau \) is the control torque (in body axes). In body axes, the star’s direction is not fixed, but rotates as

\[
\dot{\vec{d}}_0 = -\omega \times \vec{d}_0.
\]

Consider the control law

\[
\tau = -k\omega + \vec{d} \times \vec{d}_0.
\]

1. Find the equilibrium points for the spacecraft, under the given control law.
2. Consider the following function:

\[
V(\omega, \vec{d}_0) = \frac{1}{2}\omega \cdot J\omega + \frac{1}{2}|\vec{d} - \vec{d}_0|^2.
\]

Is it a good candidate for a Lyapunov function? Study the stability of the equilibrium point(s).

It may be interesting for you to simulate the system in Matlab, and look at the trajectories of the “star vector” \( \vec{d}_0 \). Notice that the control law is the sum of a “derivative” term \( k\omega \) and a “proportional” term \( \vec{d} \times \vec{d}_0 \). How do trajectories change as you choose different values of \( k \)?

**Solution:** First of all, let us write the dynamics of the system, under the given control law:

\[
\begin{align*}
J\ddot{\omega} &= -\omega \times J\omega - k\omega + \vec{d} \times \vec{d}_0, \\
\dot{\vec{d}}_0 &= -\omega \times \vec{d}_0.
\end{align*}
\]

From the second equation, we see that equilibrium points must be such that \( \omega \times \vec{d}_0 = 0 \), i.e., \( \omega \) must be parallel to \( \vec{d}_0 \), or \( \omega = c\vec{d}_0 \), for some scalar \( c \). Rewriting the first equation using this condition, we find

\[
J\ddot{\omega} = -c^2 \vec{d}_0 \times J\vec{d}_0 - ckd\vec{d}_0 + \vec{d} \times \vec{d}_0.
\]

From the above equation, we see that since \(-c^2 \vec{d}_0 \times J\vec{d}_0 + \vec{d} \times \vec{d}_0\) is always orthogonal to \( \vec{d}_0 \), \( \dot{\omega} \) can be zero only if \( c = 0 \), i.e., if \( \omega = 0 \). Moreover, we require that \( \vec{d} \times \vec{d}_0 = 0 \), i.e., \( \vec{d}_0 = \pm \vec{d} \).

Summarizing, we have two equilibria, one with the spacecraft pointing directly at the star, and one with the spacecraft pointing in the opposite direction.

The given function \( V \) is indeed a good candidate to study the stability of the “good” equilibrium \( \vec{d}_0 = \vec{d} \): it is always non-negative, and is equal to zero only at the equilibrium point \((\vec{d}_0, \omega) = (\vec{d}, 0)\). Let us study its time derivative. We get:

\[
\dot{V}(\omega, \vec{d}_0) = \omega \cdot J\omega + (\vec{d} - \vec{d}_0) \cdot \dot{\vec{d}}_0 = \omega \cdot (-\omega \times J\omega - k\omega + \vec{d} \times \vec{d}_0) + (\vec{d} - \vec{d}_0) \cdot (-\omega \times \vec{d}_0).
\]
Using the following property of the triple vector product

\[ u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u), \]

and remembering that the cross product of two parallel vector is zero, we can simplify the expression of \( \dot{V}(\vec{\omega}, \vec{d}_0) \) as

\[ \dot{V}(\vec{\omega}, \vec{d}_0) = -k|\vec{\omega}|^2, \]

i.e., \( \dot{V} \) is negative semi-definite. As a consequence, Lyapunov’s theorem only tells us that the desired equilibrium is stable in the sense of Lyapunov.

Asymptotic stability can be proven using Lasalle’s theorem, since:

- The set in which \( \dot{V} = 0 \) is the set for which \( (\vec{\omega}, \vec{d}_0) = (0, \cdot), \) i.e., any state in which the angular velocity is zero—with arbitrary attitude.
- However, in case \( \vec{d}_0 \neq \vec{d}, \) we would have \( \dot{J} \vec{\omega} \neq 0, \) thus showing that the largest invariant set in which \( \dot{V} = 0 \) is just the equilibrium \( (\vec{\omega}, \vec{d}_0) = (0, 0). \)