16.323 Final Exam

This is a closed-book exam, but you are allowed 3 pages of notes (both sides). You have 3 hours. There are six 6 questions with the relative values clearly marked.

Some handy formulas:

\[
\mathbf{u}^*(t) = \arg \left\{ \min_{\mathbf{u}(t) \in \mathcal{U}} H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \right\}
\]

\[
A^T P_{ss} + P_{ss} A + R_{xx} - P_{ss} B_u R_u^{-1} B_u^T P_{ss} = 0
\]

\[
K_{ss} = R_u^{-1} B_u^T P_{ss}
\]

\[
\lambda_i \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \frac{a + d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}
\]

(i) \( P_N = H \)

(ii) \( F_k = - \left[ R_k + B_k^T P_{k+1} B_k \right]^{-1} B_k^T P_{k+1} A_k \)

(iii) \( P_k = Q_k + F_k^T R_k F_k + \{ A_k + B_k F_k \}^T P_{k+1} \{ A_k + B_k F_k \} \)
1. (15%) Consider a model of a first order unstable plant with the state equation
\[ \dot{x}(t) = ax(t) - au(t) \quad a > 0 \]
and a new cost functional
\[ J = \int_0^\infty e^{2bt} \left[ x^2(t) + \rho u^2(t) \right] dt \quad \rho > 0, \ b \geq 0 \]
It can be shown that, with a change of variables in the system to \( e^{bt}x(t) \rightarrow \dot{x}(t) \) and \( e^{bt}u(t) \rightarrow \ddot{u}(t) \), then the only real modification required to solve for the static LQR controller with this cost functional is to use \( A + bI \) in the algebraic Riccati equation instead of \( A \).
(a) Given this information, determine the LQR gain \( K \) as a function of \( \rho, a, b \).
(b) Determine the location of the closed-loop poles for \( 0 < \rho < \infty \)
(c) Use these results to explain what the primary effect of \( b \) is on this control design.

2. (15%) Given the optimal control problem for a scalar nonlinear system:
\[ \dot{x} = xu \quad x(0) = 1 \]
\[ J(u) = x(1)^2 + \int_0^1 (x(t)u(t))^2 dt \]
find the optimal feedback strategy by solving the associated HJB equation. Hint: show that the HJB differential equation admits a solution of the form \( J^* = p(t)x(t)^2 \).

3. (15%) For the system given by the dynamics:
\[ \begin{align*}
\dot{x}_1 &= x_2 + u \\
\dot{x}_2 &= u
\end{align*} \]
with \( |u(t)| \leq 1 \), find the time optimal controller that drives the system to the origin \( x_1 = x_2 = 0 \). In your solution, clearly state:
- The switching law,
- Show the switching lines,
- Sketch the system response for \( x_1(0) = x_2(0) = 1 \).
4. (15%) Consider the following discrete system:

\[ x_{k+1} = x_k - 0.4x_k^2 + u_k \]

Assume that the state space is quantized to be \((0, 0.5, 1)\), and the control is quantized to be \((-0.4, -0.2, 0, 0.2, 0.4)\). The cost to be minimized is

\[ J = 4|x_2| + \sum_{k=0}^{1} |u_k| \]

Use dynamic programming to find the optimal control sequence and complete the following tables. If in the process you find a state that is not at the quantized value, assign it the nearest quantized value.

<table>
<thead>
<tr>
<th>(x_0)</th>
<th>(J_{0,2}^*(x_0))</th>
<th>(u_0^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
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<tr>
<td>0.5</td>
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<td></td>
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<tr>
<td>1.0</td>
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</tbody>
</table>

Use these results to find the optimal control sequence if the initial state is \(x_0 = 1\).

5. (20%) Optimal control of linear systems:

(a) Given the linear system

\[ \dot{x} = -x + u \]

with \(x(0) = 1\) and \(x(1) = 1\), find the controller that optimizes the cost functional

\[ J = \frac{1}{2} \int_0^1 (3x^2 + u^2)dt \]

Please solve this problem by forming the Hamiltonian, solve the differential equations for the state/costate, and then use these to provide the control law, which you should write as an explicit function of time.

(b) Consider the following system:

\[ \begin{align*}
\dot{x}_1 &= -x_1 + u \\
\dot{x}_2 &= -2x_2 \\
\end{align*} \]

with \(J = \int_0^\infty (x_2^2(t) + u^2(t))dt\)

- Comment on the stabilizability and detectability of this system.
- Find the optimal steady state regulator feedback law for the system. Why does this answer make sense?
- Given \(x(0) = [1 \ 1]\) what is the minimum value of the initial cost?
6. (20%) LQG control for an unstable system: Consider the unstable second order system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_2 + u + w
\end{align*}
\]

with the continuous measurements

\[y = x_1 + v\]

where \(w\) and \(v\) are zero-mean white noise processes with spectral densities \(R_{ww}\) and \(R_{vv}\) respectively and the performance index is

\[
J = \int_0^\infty (R_{xx}x_1^2 + R_{uu}u^2) \, dt
\]

You analyzed this system in Homework #3 and showed that for \(R_{xx}/R_{uu} = 1\) the steady-state LQR gains are \(K = \begin{bmatrix} 1 & \sqrt{3} + 1 \end{bmatrix}\) and the closed-loop poles are at \(s = -(\sqrt{3} \pm j)/2\).

(a) Sketch by hand the locus of the estimation error poles versus the ratio \(R_{ww}/R_{vv}\) for the steady-state LQE case. Show the pole locations for the noisy sensor problem.

(b) For \(R_{ww}/R_{vv} = 1\) show analytically that the steady-state LQE gains are

\[
L = \begin{bmatrix} \sqrt{3} + 1 \\ \sqrt{3} + 2 \end{bmatrix}
\]

and that the closed-loop poles are at \(s = -(\sqrt{3} \pm j)/2\).

(c) Find the transfer function of the corresponding steady state LQG compensator.

(d) As best as possible, provide a classical explanation of this compensator and explain why it is a good choice for this system.