This is a closed-book exam, but you are allowed 1 page of notes (both sides).

You have 1.5 hours.

There are three 3 questions of equal value.

**Hint:** To maximize your score, initially give a brief explanation of your approach before getting too bogged down in the equations.
1. For the following cost function, \( F = x^2 + y^2 - 6xy - 4x - 5y \)

(a) Minimize the cost subject to the constraints,

\[
\begin{align*}
    f_1 & : -2x + y + 1 \geq 0 \\
    f_2 & : x + y - 4 \leq 0
\end{align*}
\]

(b) How is the optimal cost affected if the constraint \( f_1 \) is changed to,

\[
    f'_1 = -2x + y + 1.1 \geq 0
\]

Estimate this difference and explain your answer.

2. The first order discrete system,

\[
x_{k+1} = x_k + u_k
\]

is to be transferred to the origin in two stages \( (x_2 = 0) \). The performance measure to be minimized is,

\[
    J = \sum_{k=0}^{1} (|x_k| + 5|u_k|)
\]

The possible state and control values are:

\[
\begin{align*}
    x_k & \in \{ 3, 2, 1, 0, -1, -2, -3 \} \\
    u_k & \in \{ 2, 1, 0, -1, -2 \}
\end{align*}
\]

(a) Use dynamic programming to determine the optimal control law and the associated cost for each possible value of \( x_0 \).

(b) Use the results from (a) to determine the optimal control sequence \( \{ u_0^*, u_1^* \} \) for the initial state \( x_0 = -2 \).
3. Consider a disturbance rejection problem that minimizes:

$$J = \frac{1}{2} x(t_f)^T H x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} x^T(t) R_{xx}(t) x(t) + u(t)^T R_{uu}(t) u(t) \ dt$$

(1)

subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t).$$

(2)

To handle the disturbance term, the optimal control should consist of both a feedback term and a feedforward term (assume \(w(t)\) is known).

$$u^*(t) = -K(t)x(t) + u_{fw}(t),$$

(3)

Using the Hamilton-Jacobi-Bellman equation, show that a possible optimal value function is of the form

$$J^*(x(t), t) = \frac{1}{2} x^T(t)P(t)x(t) + b^T(t)x(t) + \frac{1}{2}c(t),$$

(4)

where

$$K(t) = R_{uu}^{-1}(t)B^T(t)P(t), \quad u_{fw} = -R_{uu}^{-1}(t)B^T(t)b(t)$$

(5)

In the process demonstrate that the conditions that must be satisfied are:

$$
\begin{align*}
-\dot{P}(t) &= A^T(t)P(t) + P(t)A(t) + R_{xx}(t) - P(t)B(t)\left[R_{uu}^{-1}(t)B^T(t)P(t)\right] \\
\dot{b}(t) &= -\left[A(t) - B(t)R_{uu}^{-1}(t)B^T(t)P(t)\right] b(t) - P(t)w(t) \\
\dot{c}(t) &= b^T(t)B(t)R_{uu}^{-1}(t)B^T(t)b(t) - 2b^T(t)w(t).
\end{align*}
$$

with boundary conditions: \(P(t_f) = H, \ b(t_f) = 0, \ c(t_f) = 0.\)