16.323 Lecture 10

Singular Arcs

- Bryson Chapter 8
- Kirk Section 5.6
Singular Problems

- There are occasions when the PMP

\[ u^*(t) = \text{arg} \min_{u(t) \in \mathcal{U}} H(x, u, p, t) \]

fails to define \( u^*(t) \Rightarrow \) can an extremal control still exist?
- Typically occurs when the Hamiltonian is linear in the control, and the coefficient of the control term equals zero.

- Example: on page 9-10 we wrote the control law:

\[
\begin{cases}
-u_m & b < p_2(t) \\
0 & -b < p_2(t) < b \\
u_m & p_2(t) < -b
\end{cases}
\]

but we do not know what happens if \( p_2 = b \) for an interval of time.
- Called a singular arc.
- Bottom line is that the straightforward solution approach does not work, and we need to investigate the PMP conditions in more detail.

- Key point: depending on the system and the cost, singular arcs might exist, and we must determine their existence to fully characterize the set of possible control solutions.

- Note: control on the singular arc is determined by the requirements that the coefficient of the linear control terms in \( H_u \) remain zero on the singular arc and so must the time derivatives of \( H_u \).
- Necessary condition for scalar \( u \) can be stated as

\[
(-1)^k \frac{\partial}{\partial u} \left[ \left( \frac{d^{2k}}{dt^{2k}} \right) H_u \right] \geq 0 \quad k = 0, 1, 2 \ldots
\]
Singular Arc Example 1

• With $\dot{x} = u$, $x(0) = 1$ and $0 \leq u(t) \leq 4$, consider objective

$$\min \int_0^2 (x(t) - t^2)^2 dt$$

• First form standard Hamiltonian

$$H = (x(t) - t^2)^2 + p(t)u(t)$$

which gives $H_u = p(t)$ and

$$\dot{p}(t) = -H_x = -2(x - t^2), \quad \text{with } p(2) = 0 \quad (10.15)$$

• Note that if $p(t) > 0$, then PMP indicates that we should take the minimum possible value of $u(t) = 0$.
  - Similarly, if $p(t) < 0$, we should take $u(t) = 4$.

• Question: can we get that $H_u \equiv 0$ for some interval of time?
  - Note: $H_u \equiv 0$ implies $p(t) \equiv 0$, which means $\dot{p}(t) \equiv 0$, and thus

$$\dot{p}(t) \equiv 0 \Rightarrow x(t) = t^2, \quad u(t) = \dot{x} = 2t$$

• Thus we get the control law that

$$u(t) = \begin{cases} 0 & \text{when } p(t) > 0 \\ 2t & \text{when } p(t) = 0 \\ 4 & \text{when } p(t) < 0 \end{cases}$$
• Can show by contradiction that optimal solution has $x(t) \geq t^2$ for $t \in [0, 2]$.
  - And thus we know that $\dot{p}(t) \leq 0$ for $t \in [0, 2]$.
  - But $p(2) = 0$ and $\dot{p}(t) \leq 0$ imply that $p(t) \geq 0$ for $t \in [0, 2]$.

• So there must be a point in time $k \in [0, 2]$ after which $p(t) = 0$ (some steps skipped here...)
  - Check options: $k = 0$? $\Rightarrow$ contradiction
  - Check options: $k = 2$? $\Rightarrow$ contradiction

• So must have $0 < k < 2$. How find it? Control law will be
  
  $u(t) = \begin{cases} 
  0 & \text{when } 0 \leq t < k \\
  2t & \text{when } k \leq t < 2
  \end{cases}$

  apply this control to the state equations and get:
  
  $x(t) = \begin{cases} 
  1 & \text{when } 0 \leq t \leq k \\
  t^2 + (1 - k^2) & \text{when } k \leq t \leq 2
  \end{cases}$

  To find $k$, note that must have $p(t) \equiv 0$ for $t \in [k, 2]$, so in this time range
  
  $\dot{p}(t) \equiv 0 = -2(1 - k^2) \Rightarrow k = 1$

  - So now both $u(t)$ and $x(t)$ are known, and the optimal solution is to “bang off” and then follow a singular arc.
Singular Arc Example 2

- LTI system, \( x_1(0), x_2(0), t_f \) given; \( x_1(t_f) = x_2(t_f) = 0 \)
  \[
  A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
  \]
  and \( J = \frac{1}{2} \int_0^{t_f} x_1^2 dt \) (see Bryson and Ho, p. 248)

- So \( H = \frac{1}{2} x_1(t)^2 + p_1(t)x_2(t) + p_1(t)u(t) - p_2(t)u(t) \)
  \[\Rightarrow \dot{p}_1(t) = -x_1(t), \quad \dot{p}_2(t) = -p_1(t)\]

- For a singular arc, we must have \( H_u = 0 \) for a finite time interval
  \( H_u = p_1(t) - p_2(t) = 0? \)

- Thus, during that interval
  \[
  \frac{d}{dt} H_u = \dot{p}_1(t) - \dot{p}_2(t) = -x_1(t) + p_1(t) = 0
  \]

- Note that \( H \) is not an explicit function of time, so \( H \) is a constant for all time
  \[
  H = \frac{1}{2} x_1(t)^2 + p_1(t)x_2(t) + [p_1(t) - p_2(t)] u(t) = C
  \]
  but can now substitute from above along the singular arc
  \[
  \frac{1}{2} x_1(t)^2 + x_1(t)x_2(t) = C
  \]
  which gives a family of singular arcs in the state \( x_1, x_2 \)

- To find the appropriate control to stay on the arc, use
  \[
  \frac{d^2}{dt^2} (H_u) = -\dot{x}_1 + \dot{p}_1 = - (x_2(t) + u(t)) - x_1(t) = 0
  \]
  or that \( u(t) = -(x_1(t) + x_2(t)) \) which is a linear feedback law to use along the singular arc.
Consider the min time-fuel problem for the general system

\[ \dot{x} = Ax + Bu \]

with \( M^- \leq u_i \leq M^+ \) and

\[ J = \int_0^{t_f} (1 + \sum_{i=1}^m c_i|u_i|) dt \]

\( t_f \) is free and we want to drive the state to the origin

- We studied this case before, and showed that

\[ H = 1 + \sum_{i=1}^m (c_i|u_i| + p^T B_i u_i) + p^T A x \]

- On a singular arc, \( \frac{d^k}{dt^k} (H u) = 0 \Rightarrow \) coefficient of \( u \) in \( H \) is zero

\[ \Rightarrow p^T(t) B_i = \pm c_i \]

for non-zero period of time and

\[ \frac{d^k}{dt^k} (p^T(t) B_i) = \left( \frac{d^k p(t)}{dt^k} \right)^T B_i = 0 \quad \forall \ k \geq 1 \]

- Recall the necessary conditions \( \dot{p}^T = -Hx = -p^T A \), which imply

\[ \ddot{p}^T = -p^T A = p^T A^2 \]

\[ \dddot{p}^T = -p^T A = -p^T A^3 \]

\[ \vdots \]

\[ \left( \frac{d^k p(t)}{dt^k} \right)^T \equiv (-1)^k p^T A^k \]

and combining with the above gives

\[ \left( \frac{d^k p(t)}{dt^k} \right)^T B_i = (-1)^k p^T A^k B_i = 0 \]
• Rewriting these equations yields the conditions that
\[ p^T A B_i = 0, \quad p^T A^2 B_i = 0, \quad \ldots \]
\[ \Rightarrow p^T A \left[ B_i \ AB_i \ \cdots \ A^{n-1}B_i \right] = 0 \]

• There are three ways to get:
\[ p^T A \left[ B_i \ AB_i \ \cdots \ A^{n-1}B_i \right] = 0 \]

• On a singular arc, we know that \( p(t) \neq 0 \) so this does not cause the condition to be zero.

• What if \( A \) singular, and \( p(t)^T A = 0 \) on the arc?
  – Then \( \dot{p}^T = -p^T A = 0 \). In this case, \( p(t) \) is constant over \([t_0, t_f]\)
  – Indicates that if the problem is singular at any time, it is singular for all time.
  – This also indicates that \( u \) is a constant.
  – A possible case, but would be unusual since it is very restrictive set of control inputs.

• Third possibility is that \( \left[ B_i \ AB_i \ \cdots \ A^{n-1}B_i \right] \) is singular, meaning that the system is not controllable by the individual control inputs.
  – Very likely scenario – most common cause of singularity conditions.
  – Lack of controllability by a control input does not necessarily mean that a singular arc has to exist, but it is a possibility.
• For **Min Time** problems, now $c_i = 0$, so things are a bit different

• In this case the switchings are at $p^T B_i = 0$ and a similar analysis as before gives the condition that

$$p^T \begin{bmatrix} B_i & AB_i & \cdots & A^{n-1}B_i \end{bmatrix} = 0$$

• Now there are only 2 possibilities
  – $p = 0$ is one, but in that case,
    $$H = 1 + p^T (Ax + Bu) = 1$$
    but we would expect that $H = 0$
  – Second condition is obviously the lack of controllability again.

• **Summary (Min time):**
  – If the system is completely controllable by $B_i$, then $u_i$ can have no singular intervals
  – Not shown, but if the system is not completely controllable by $B_i$, then $u_i$ **must** have a singular interval.

• **Summary (Min time-fuel):**
  – If the system is completely controllable by $B_i$ and $A$ is non-singular, then there can be no singular intervals
Consider systems that are nonlinear in the state, but linear in the control

\[ \dot{x}(t) = a(x(t)) + b(x(t))u(t) \]

with cost

\[ J = \int_{t_0}^{t_f} g(x(t)) \, dt \]

For a singular arc, in general you will find that

\[ \frac{d^k}{dt^k}(H_{u_i}) = 0 \quad k = 0, \ldots, r - 1 \]

but these conditions provide no indication of the control required to keep the system on the singular arc

– i.e. the coefficient of the control terms is zero.

But then for some \( r \) and \( i \), \( \frac{dr}{dt^r}(H_{u_i}) = 0 \) does retain \( u_i \).

– So if \( u_j(x, p) \) are the other control inputs, then

\[ \frac{d^r}{dt^r}(H_{u_i}) = C(x, p, u_j(x, p)) + D(x, p, u_j(x, p))u_i = 0 \]

with \( D \neq 0 \), so the condition does depend on \( u_i \).

Then can define the appropriate control law to stay on the singular arc as

\[ u_i = -\frac{C(x, p, u_j(x, p))}{D(x, p, u_j(x, p))} \]
• Properties of this solution are:
  – $r \geq 2$ is even
  – Singular surface of dimension $2n - r$ in space of $(x, p)$ in general, but $2n - r - 1$ if $t_f$ is free (additional constraint that $H(t) = 0$)
  – Additional necessary condition for the singular arc to be extremal is that:
    $$(-1)^{r/2} \frac{\partial}{\partial u_i} \left[ \frac{d^r}{dt^r} H_u \right] \geq 0$$
  – Note that in the example above,
    $$\frac{\partial}{\partial u_i} \left[ \frac{d^r}{dt^r} H_u \right] \sim D$$
Last Example

- Goddard problem: thrust program for maximum altitude of a sounding rocket [Bryson and Ho, p. 253]. Given the EOM:

\[
\begin{align*}
\dot{v} &= \frac{1}{m} [F(t) - D(v, h)] - g \\
\dot{h} &= v \\
\dot{m} &= \frac{-F(t)}{c}
\end{align*}
\]

where \( g \) is a constant, and drag model is \( D(v, h) = \frac{1}{2} \rho v^2 C_d S e^{-\beta h} \)

- Problem: Find \( 0 \leq F(t) \leq F_{\text{max}} \) to maximize \( h(t_f) \) with \( v(0) = h(0) = 0 \) and \( m(0), m(t_f) \) are given

- The Hamiltonian is

\[
H = p_1 \left( \frac{1}{m} [F(t) - D(v, h)] - g \right) + p_2 v - p_3 \frac{F(t)}{c}
\]

and since \( v(t_f) \) is not specified and we are maximizing \( h(t_f) \),

\( p_2(t_f) = -1 \quad p_1(t_f) = 0 \)

- Note that \( H(t) = 0 \) since the final time is not specified.

- The costate EOM are:

\[
\dot{p} = \begin{bmatrix}
\frac{1}{m} \frac{\partial D}{\partial v} & -1 & 0 \\
\frac{1}{m} \frac{\partial D}{\partial h} & 0 & 0 \\
\frac{F-D}{m^2} & 0 & 0
\end{bmatrix} p
\]

- \( H \) is linear in the controls, and the minimum is found by minimizing \( (\frac{p_1}{m} - \frac{p_3}{c}) F(t) \), which clearly has 3 possible solutions:

\[
\begin{align*}
F &= F_{\text{max}} \quad & (\frac{p_1}{m} - \frac{p_3}{c}) < 0 \\
0 < F < F_{\text{max}} & \quad \text{if} \quad (\frac{p_1}{m} - \frac{p_3}{c}) = 0 \\
F &= 0 \quad & (\frac{p_1}{m} - \frac{p_3}{c}) > 0
\end{align*}
\]

- Middle expression corresponds to a singular arc.
• Note: on a singular arc, must have $H_u = p_1c - p_3m = 0$ for finite interval, so then $\dot{H}_u = 0$ and $\ddot{H}_u = 0$, which means

$$\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right)p_1 - mp_2 = 0$$

and

$$F = D + mg + \frac{m}{D + 2c\frac{\partial D}{\partial v} + c^2\frac{\partial^2 D}{\partial v^2}} \left[-g(D + c\frac{\partial D}{\partial v}) + c(c - v)\frac{\partial D}{\partial h} - vc^2\frac{\partial^2 D}{\partial v \partial h}\right]$$

which is a nonlinear feedback control law for thrust on a singular arc.

– For this particular drag model, the feedback law simplifies to:

$$F = D + mg + \frac{mg}{1 + 4(c/v) + 2(c/v)^2} \left[\frac{\beta c^2}{g} \left(1 + \frac{v}{c}\right) - 1 - 2\frac{c}{v}\right]$$

and the singular surface is: $mg = \left(1 + \frac{v}{c}\right)D$

• Constraints $H(t) = 0$, $H_u = 0$, and $\dot{H}_u = 0$ provide a condition that defines a surface for the singular arc in $v, h, m$ space:

$$D + mg - \frac{v}{c}D - v\frac{\partial D}{\partial v} = 0$$

(10.17)

• It can then be shown that the solution typically consists of 3 arcs:

1. $F = F_{\text{max}}$ until 10.17 is satisfied.
2. Follow singular arc using 10.16 feedback law until $m(t) = m(t_f)$.
3. $F = 0$ until $v = 0$.

which is of the form “bang-singular-bang”
Figure 10.1: Goddard Problem
Figure 10.2: Goddard Problem
Figure 10.3: Goddard Problem