Topic #14

16.31 Feedback Control Systems

MIMO Systems
- Singular Value Decomposition
Multivariable Frequency Response

- In the MIMO case, the system $G(s)$ is described by a $p \times m$ transfer function matrix (TFM)
  - Still have that $G(s) = C(sI - A)^{-1}B + D$
  - But $G(s) \rightarrow A, B, C, D$ MUCH less obvious than in SISO case.
  - Also seen that the discussion of poles and zeros of MIMO systems is much more complicated.

- In SISO case we use the Bode plot to develop a measure of the system “size”.
  - Given $z = Gw$, where $G(j\omega) = |G(j\omega)|e^{j\phi(w)}$
  - Then $w = |w|e^{j(\omega_1t + \psi)}$ applied to $|G(j\omega)|e^{j\phi(w)}$ yields $|w||G(j\omega_1)|e^{j(\omega_1t + \psi_0 + \phi(\omega_1))} = |z|e^{j(\omega_1t + \psi_0)} \equiv z$
  - Amplification and phase shift of the input signal obvious in the SISO case.

- MIMO extension?
  - Is the response of the system large or small?
    \[
    G(s) = \begin{bmatrix}
      10^3/s & 0 \\
      0 & 10^{-3}/s
    \end{bmatrix}
    \]
• For MIMO systems, cannot just plot all of the $g_{ij}$ elements of $G$
  – Ignores the coupling that might exist between them.
  – So not enlightening.

• **Basic MIMO frequency response:**
  – Restrict all inputs to be at the same frequency
  – Determine how the system responds at that frequency
  – See how this response changes with frequency

• So inputs are $w = wc e^{j\omega t}$, where $w_c \in \mathbb{C}^m$
  – Then we get $z = G(s)|_{s=j\omega} \cdot w_c$, \( \Rightarrow z = z_c e^{j\omega t} \) and $z_c \in \mathbb{C}^p$
  – We need only analyze $z_c = G(j\omega)w_c$

• As in the SISO case, we need a way to establish if the system response is **large** or **small**.
  – How much amplification we can get with a bounded input.

• Consider $z_c = G(j\omega)w_c$ and set $\|w_c\|_2 = \sqrt{w_c^H w_c} \leq 1$. What can we say about the $\|z_c\|_2$?
  – Answer depends on $\omega$ and on the **direction** of the input $w_c$
  – Best found using **singular values**.
Singualr Value Decomposition

- Must perform SVD of the matrix \( G(s) \) at each frequency \( s = j\omega \)

\[
G(j\omega) \in \mathbb{C}^{p \times m} \quad U \in \mathbb{C}^{p \times p} \quad \Sigma \in \mathbb{R}^{p \times m} \quad V \in \mathbb{C}^{m \times m}
\]

\[
G = U\Sigma V^H
\]

- \( U^HU = I, \quad UU^H = I, \quad V^HV = I, \quad VV^H = I \), and \( \Sigma \) is diagonal.

- Diagonal elements \( \sigma_k \geq 0 \) of \( \Sigma \) are the singular values of \( G \).

\[
\sigma_i = \sqrt{\lambda_i(G^HG)} \quad \text{or} \quad \sigma_i = \sqrt{\lambda_i(GG^H)}
\]

the positive ones are the same from both formulas.

- Columns of matrices \( U \) and \( V \) \( (u_i \text{ and } v_j) \) are the associated eigenvectors

\[
G^Hv_j = \sigma_j^2v_j
\]

\[
GG^Hu_i = \sigma_i^2u_i
\]

\[
Gv_i = \sigma_iu_i
\]

- If the rank\( (G) = r \leq \min(p,m) \), then
  - \( \sigma_k > 0, \quad k = 1, \ldots, r \)
  - \( \sigma_k = 0, \quad k = r + 1, \ldots, \min(p,m) \)
  - Singular values are sorted so that \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \)

- An SVD gives a very detailed description of how a matrix (the system \( G \)) acts on a vector (the input \( \mathbf{w} \)) at a particular frequency.
• So how can we use this result?
  – Fix the size $\|w_c\|_2 = 1$ of the input, and see how large we can make the output.
  – Since we are working at a single frequency, we just analyze the relation
    $$z_c = G_w w_c, \quad G_w \equiv G(s = j\omega)$$

• Define the maximum and minimum amplifications as:
  $$\bar{\sigma} \equiv \max_{\|w_c\|_2=1} \|z_c\|_2$$
  $$\underline{\sigma} \equiv \min_{\|w_c\|_2=1} \|z_c\|_2$$

• Then we have that (let $q = \min(p, m)$)
  $$\bar{\sigma} = \sigma_1$$
  $$\underline{\sigma} = \begin{cases} 
    \sigma_q & p \geq m \quad \text{“tall”} \\
    0 & p < m \quad \text{“wide”} 
  \end{cases}$$

• Can use $\bar{\sigma}$ and $\underline{\sigma}$ to determine the possible amplification and attenuation of the input signals.

• Since $G(s)$ changes with frequency, so will $\bar{\sigma}$ and $\underline{\sigma}$
SVD Example

- Consider (wide case)

\[
G_w = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \Sigma V^H
\]

so that \( \sigma_1 = 5 \) and \( \sigma_2 = 0.5 \)

\[
\bar{\sigma} \equiv \max_{\|w_c\|_2=1} \|G_w w_c\|_2 = 5 = \sigma_1
\]

\[
\sigma \equiv \min_{\|w_c\|_2=1} \|G_w w_c\|_2 = 0 \neq \sigma_2
\]

- But now consider (tall case)

\[
\tilde{G}_w = \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U \Sigma V^H
\]

so that \( \sigma_1 = 5 \) and \( \sigma_2 = 0.5 \) still.

\[
\bar{\sigma} \equiv \max_{\|w_c\|_2=1} \|G_w w_c\|_2 = 5 = \sigma_1
\]

\[
\sigma \equiv \min_{\|w_c\|_2=1} \|G_w w_c\|_2 = 0.5 = \sigma_2
\]
• For MIMO systems, the gains (or \( \sigma \)'s) are only part of the story, as we must also consider the **input direction**.

• To analyze this point further, note that we can rewrite

$$G_w = U \Sigma V^H = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix} \begin{bmatrix} v_1^H \\ \vdots \\ v_m^H \end{bmatrix}$$

= \sum_{i=1}^{m} \sigma_i u_i v_i^H

– Assumed tall case for simplicity, so \( p > m \) and \( q = m \)

• Can now analyze impact of various alternatives for the input

  – Only looking at one frequency, so the basic signal is harmonic.

  – But, we are free to pick the relative **sizes** and **phases** of each of the components of the input vector \( w_c \).

  ◦ These define the **input direction**
• For example, we could pick $w_c = v_1$, then

$$z_c = G_w w_c = \left( \sum_{i=1}^{m} \sigma_i u_i v_i^H \right) v_1 = \sigma_1 u_1$$

since $v_i^H v_j = \delta_{ij}$.

– Output amplified by $\sigma_1$. The relative sizes and phases of each of the components of the output are given by the vector $z_c$.

• By selecting other input directions (at the same frequency), we can get quite different amplifications of the input signal

$$\sigma \leq \frac{\|G_w w_c\|_2}{\|w_c\|_2} \leq \sigma$$

• Thus we say that
  – $G_w$ is large if $\sigma(G_w) \gg 1$
  – $G_w$ is small if $\sigma(G_w) \ll 1$

• **MIMO frequency response** are plots of $\sigma(j\omega)$ and $\sigma(j\omega)$.
  – Then use the singular value vectors to analyze the response at a particular frequency.